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VIBRATIONS AND STABILITY OF BUCKLED RECTANGULAR PLATES

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CHAPTER I

INTRODUCTION

Current structural design practice calls for decreasing structural thickness as a result of weight limitations; as a consequence, many structures are permitted to buckle and are then used in the post-buckled state. Members that previously served only in a nonstructural capacity are used to sustain loads greater than those predicted in the usual "Euler load" sense. In addition, structures subjected to these high static loads are frequently expected to survive dynamic disturbances. This is particularly true in aircraft and space structures where the stiffness and dynamic characteristics of a buckled rectangular panel have become important with increasing flight speeds. The buckling of the skin panels, whether caused by air loads or by thermal expansion, will cause a marked reduction in the stiffness of the structure. The changes in frequencies and mode shapes that take place as a result of thermal expansion affect the various static and dynamic instabilities considerably. The purpose of the present study is to determine the dynamic characteristics, that is, the natural frequencies and mode shapes of vibration of a rectangular plate, in terms of a load parameter both before and after buckling.

The free vibrations of elastic bodies or structures about the unbuckled equilibrium configuration have been studied extensively before. The natural frequency and the mode shape of vibration are obtained from the solution of an eigenvalue problem. If such a body or structure is first preloaded statically, then the resulting frequency of vibration is increased by tensile stresses or forces and decreased by compressive forces. In the case of compressive loading, it goes to zero when the compressive force reaches the buckling load.

The most familiar example of such a problem is the lateral vibration of a simply supported bar which is axially loaded. The square of the frequency of the vibration is linearly related to the axial force. Willers(1) has calculated the decrease in the natural frequency of a clamped circular plate under uniform radial compressive forces. Massonnet(2) and Lurie(3) have shown the existence of an intimate relationship between normal vibrations and instability. A definitive discussion can be found in the book by Bolotin.(4) In general, within the framework of linear theories, whenever the mode shape of buckling and of vibration in the presence of axial load are the same, the square of the natural frequency varies linearly with increasing axial load until it vanishes at the corresponding buckling load. This property is often used to predict the buckling load by extrapolation of a few points obtained experimentally at relatively low loads on the frequency squared-load curve.

The buckling of a simply supported plate under edge compression was first studied by Bryan(5) in 1891. The buckling loads for plates that are not simply supported have been investigated extensively by Timoshenko.(6) These problems are all solved within the framework of linear classical theory under the assumption that the deflection of the plate is small in comparison with its thickness; therefore the solution applies only to the incipient state of buckling. It is obvious that the linear theory of plates no longer applies when the behavior of the plate above the buckling load is to be investigated.

A set of nonlinear differential equations for plates with large deflections was introduced in 1910 by von Kármán.[7] Owing to the nonlinearity of the equations, there exist relatively few exact solutions. However, various approximate solutions have been presented by Cox[8] and Timoshenko,[6] and a more accurate solution of the problem of large deflections has been given by Marguerre.[9] By means of Fourier series Levy[10] has obtained an "exact" solution to the large deflection equations of von Kármán for square plates. Friedrichs and Stoker[11,12] have used methods of perturbation, power series and asymptotic expansions to solve, in a very exhaustive manner, the problem of a simply supported circular plate subjected to radial compressive loading. Alexeev,[13] using a method of successive approximations, has obtained a solution for the square plate buckling into both one buckle and two buckles. Masur[14] has utilized a stress function space together with a minimum energy principle to obtain a sequence of solutions with error estimates for the post-buckling behavior of plates. With the exception of the analysis of Alexeev,[13] all of the above studies of the post-buckling behavior of plates are concerned with primary buckling.

Secondary buckling has been observed through experiments,[15,16,17] and in the case of circular plates, the instability of the primary buckling mode has been pointed out by several authors.[11,14] Further, Stein[18] has used a perturbation technique to convert the nonlinear large deflection equations of von Kármán into a set of linear equations and to investigate the post-buckling behavior of simply supported rectangular plates by solving the first few of the equations. His investigation indicates possible changes in buckle pattern; the same has also been noted by Koiter.[19]

Bisplinghoff and Pian[20] have treated the case of vibration of a thermally stressed rectangular plate which is simply supported and free to displace laterally. Shulman[21] has considered the case of a uniformly heated plate with two opposite edges simply supported and with generalized support conditions on the other two edges. Both papers consider the small vibrations of the plate in its pre- and post-buckling states, the analysis of the latter being approximate. Herzog and Masur[22] have treated the case of vibration of a buckled circular plate by means of both perturbation techniques and power series expansions. Their analysis is "exact" within the limits of classical plate theory, small amplitude vibration and in the sense of a converging series which has been truncated.

The present study is concerned with the linearized vibrations of a rectangular plate relative to a static buckled configuration, and with the instability of the buckling modes. Both the static and dynamic equations of equilibrium are solved by perturbation techniques. If perturbation coefficients up to the third order are included, the results are acceptable for a significant range of the loading parameter. For large values of the latter the frequency of vibration of the plate is obtained by means of the Galerkin method while the static problem is solved by a method similar to the one due to Marguerre.

CHAPTER II

FORMULATION OF THE PROBLEM

In what follows we consider the xy plane to be the middle plane of an elastic, isotropic plate and z the direction of the lateral deflection. The plate is subjected to membrane forces in the plane of the plate. For the sake of convenience, the index notation is used for the general discussion of the problem, with Latin subscripts i, j and k taking the values of x and y, a repeated subscript representing the sum of all allowable values of that subscript, and a comma followed by a Latin subscript denoting appropriate differentiation.

Let a plate of thickness h be subjected to prescribed edge thrusts λT_i on B' and to displacements λU_i on B'', in which $B = B' + B''$ forms the boundary of the region R of the middle plane and λ is a parameter assuming increasing positive values. The membrane displacements and stresses u_i and t_{ij}, respectively, may then be conveniently characterized by

$$\begin{aligned} u_i &= \lambda u_i^o + U_i' \\ t_{ij} &= \lambda t_{ij}^o + T_{ij}' \end{aligned} \tag{2.1}$$

In Equations (2.1) the first terms on the right side correspond to the unbuckled state and are governed by the customary "generalized plane stress" equations

$$\left.\begin{aligned} t_{ij}^o &= \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(u_{i,j}^o + u_{j,i}^o) + \nu u_{k,k}^o \delta_{ij}\right] = t_{ji}^o \\ t_{ij,j}^o &= 0 \end{aligned}\right\} \quad \text{in } R \tag{2.2}$$

$$t^o_{ij} n_j = T_i \quad \text{on } B' \qquad u^o_i = U_i \quad \text{on } B'' \tag{2.3}$$

in which E and ν are Young's modulus and Poisson's ratio, respectively, δ_{ij} is the Kronecker delta, and n_i are the components of the unit outer normal.

The second (primed) terms in Equations (2.1) represent the changes induced by buckling and satisfy the set of equations

$$T'_{ij} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(U'_{i,j} + U'_{j,i} + W_{,i}W_{,j}) + \nu(U'_{k,k} + \tfrac{1}{2}W_{,k}W_{,k})\delta_{ij}\right] = T'_{ji} \qquad T'_{ij,j} = 0 \qquad \text{in } R \tag{2.4}$$

$$T'_{ij} n_j = 0 \quad \text{on } B' \qquad U'_i = 0 \quad \text{on } B'' \tag{2.5}$$

in which the static deflection W satisfies the additional equation

$$D\Delta\Delta W - h(\lambda t^o_{ij} + T'_{ij})W_{,ij} = 0 \quad \text{in } R \tag{2.6}$$

and appropriate boundary conditions on B. In Equation (2.6) Δ stands for the Laplacian operator and D, the bending stiffness, is given by

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{2.7}$$

The separation of the solution into two parts in line with Equation (2.1) has been found convenient because of the linear homogeneity of Equations (2.4) and (2.5) in U'_i and T'_{ij}. That is, for a given function $W(x,y)$ these equations represent a boundary value problem whose solution may be expressed symbolically by means of

$$T'_{ij} = \frac{1}{2} \left\langle W_{,i}W_{,j}\right\rangle \tag{2.8}$$

The operator so defined obeys appropriate superposition principles, e.g.,

$$\langle W^1_{,i}+W^2_{,i}, W^1_{,j}+W^2_{,j}\rangle = \langle W^1_{,i}W^1_{,j}\rangle + \langle W^2_{,i}W^1_{,j}\rangle + \langle W^1_{,i}W^2_{,j}\rangle + \langle W^2_{,i}W^2_{,j}\rangle \qquad (2.9)$$

It is also noted that for sufficiently small values of λ (say, $\lambda \leqq \lambda_o$), Equations (2.4), (2.5), and (2.6) admit only trivially vanishing solutions. For $\lambda > \lambda_o$ these represent unstable configurations. Other (i.e. buckled) configurations exist in that case, although not all of these may be stable.

If a small vibration $w(x,y)e^{i\omega t}$ is superimposed on W, then, after linearization with respect to w, the governing equation of motion is

$$D\Delta\Delta w - h(\lambda t^o_{ij} + T'_{ij})w_{,ij} - ht'_{ij}W_{,ij} - h\mu w = 0 \qquad (2.10)$$

in which

$$\mu = \rho\omega^2 \qquad (2.11)$$

with ρ representing the mass density. The dynamic membrane stress t'_{ij} (or rather its amplitude) is given symbolically by

$$t'_{ij} = \frac{1}{2}\langle W_{,i}w_{,j} + w_{,i}W_{,j}\rangle \qquad (2.12)$$

if in-plane inertia is ignored.*

We consider now a rectangular, simply supported plate covering the region $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. It is postulated that the edges are made to approach one another by a specified amount and are then held fixed during the vibration. This seemingly artificial type of boundary

*For the case of shallow shells this has been justified in Ref. 23.

condition is equivalent to fixing* the boundary while the plate is heated uniformly; this is considered to be a reasonably realistic representation of actual conditions.

The complete set of boundary conditions for the static case is therefore as follows:

$$B_1(W) \equiv W(0,y) = W(a,y) = W(x,0) = W(x,b) = 0 \tag{2.13}$$

$$B_2(W) \equiv W_{,xx}(0,y) = W_{,xx}(a,y) = W_{,yy}(x,0) = W_{,yy}(x,b) = 0 \tag{2.14}$$

$$\begin{aligned} &u^o(0,y) = v^o(x,0) = 0;\ u^o(a,y) = U_E,\ v^o(x,b) = V_E \\ &v^o_{,x}(0,y) = v^o_{,x}(a,y) = u^o_{,y}(x,0) = u^o_{,y}(x,b) = 0 \end{aligned} \tag{2.15}$$

$$\begin{aligned} &U'(0,y) = U'(a,y) = V'(x,0) = V'(x,b) = 0 \\ &V'_{,x}(0,y) = V'_{,x}(a,y) = U'_{,y}(x,0) = U'_{,y}(x,b) = 0 \end{aligned} \tag{2.16}$$

U_E and V_E are the magnitude of the displacements which are required to cause the plate to buckle in the linear sense; thus the value of λ determines the extent to which the critical deformation (or temperature increase) has been exceeded.

For the dynamic case the boundary conditions are

$$B_1(w) \equiv w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \tag{2.17}$$

$$B_2(w) \equiv w_{,xx}(0,y) = w_{,xx}(a,y) = w_{,yy}(x,0) = w_{,yy}(x,b) = 0 \tag{2.18}$$

$$\begin{aligned} &u'(0,y) = u'(a,y) = v'(x,0) = v'(x,b) = 0 \\ &v'_{,x}(0,y) = v'_{,x}(a,y) = u'_{,y}(x,0) = u'_{,y}(x,b) = 0 \end{aligned} \tag{2.19}$$

in which u' and v' are the dynamic displacement amplitudes of a point in the x and y directions, respectively.

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* Actually, fixity is assumed only in the normal direction, while the plate is free to slide in the direction of the boundary. This type of shearless constraint reduces the computational labor enormously, yet is believed to introduce no significant deviation from the computationally far more intractable condition of full fixity.

CHAPTER III

THE PERTURBATION SOLUTION

In this chapter we obtain a solution to both the static and dynamic problem through a perturbation expansion. As usual this method is operative only within a limited range of the perturbation parameter; for large values the series converges too slowly to be handled without excessive labor. In the present case the results obtained appear to be acceptable up to a value of at least ten of the post-buckling parameter λ. The static portion is similar to previous work by Stein,[18] but has had to be rederived in order to make the dynamic portion comprehensible.

We consider first the static case. It is required to solve Equation (2.6), in which t^o_{ij} and u^o_i satisfy Equations (2.2) and T'_{ij}, U'_i and W satisfy Equations (2.4), with the associated boundary conditions Equations (2.13), (2.14), (2.15) and (2.16).

Equations (2.2) and (2.15) represent the usual problem of plane elasticity, whose well-known solution for a rectangular plate is

$$\begin{aligned} u^o(x,y) &= U_E \frac{x}{a} \\ v^o(x,y) &= V_E \frac{y}{b} \end{aligned} \tag{3.1}$$

where U_E and V_E are found later on.

We now assume the functions W and λ to be expandable in a power series in terms of an arbitrary parameter ϵ in the neighborhood of the point of buckling $\epsilon = 0$, that is, with $W \equiv W(x,y,\epsilon)$,

$$W = \epsilon W^{(1)} + \epsilon^3 W^{(3)} + \epsilon^5 W^{(5)} + \dots \tag{3.2}$$

$$\lambda = \lambda_o + \epsilon^2\lambda_2 + \epsilon^4\lambda_4 + \dots . \tag{3.3}$$

Here $W^{(n)}$* are functions of x,y only and ϵ, the perturbation parameter, will be assumed to be monotonely increasing as buckling progresses. The fact that W is odd and λ is even in ϵ may be easily verified upon substitution in the relevant equations. For the sake of brevity these steps are omitted here. Since at the point of buckling, $\epsilon = 0$, λ_o is identified as the load parameter for the Euler buckling load. In view of Equations (2.8) and (2.9) T'_{ij} can be expressed in terms of the arbitrary parameter ϵ as follows:

$$T'_{ij} = \sum_{p=1}^{\infty} \epsilon^p T_{ij}^{(p)} \tag{3.4}$$

in which

$$T_{ij}^{(p)} = \frac{1}{2} \left\langle W_{,i}^{(p-1)} W_{,j}^{(1)} + W_{,i}^{(p-2)} W_{,j}^{(2)} + \ldots + W_{,i}^{(1)} W_{,j}^{(p-1)} \right\rangle \tag{3.5}$$

The membrane stress equilibrium equations can be written in terms of the additional displacements as

$$\begin{aligned} U'_{,xx} + \frac{1-\nu}{2} U'_{,yy} + \frac{1+\nu}{2} V'_{,xy} &= -W_{,x}W_{,xx} - \frac{1+\nu}{2} W_{,y}W_{,xy} - \frac{1-\nu}{2} W_{,x}W_{,yy} \\ V'_{,yy} + \frac{1-\nu}{2} V'_{,xx} + \frac{1+\nu}{2} U'_{,xy} &= -W_{,y}W_{,yy} - \frac{1+\nu}{2} W_{,x}W_{,xy} - \frac{1-\nu}{2} W_{,y}W_{,xx} \end{aligned} \tag{3.6}$$

In view of this the additional displacements U' and V' can also be expanded in a power series of the same arbitrary parameter ϵ, and the series is expected to start with the second power of ϵ and to contain

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* Superscripts in parentheses are intended to identify the variable and not to act as an exponent. Whenever possible, however, parentheses will be omitted where there is no possible confusion and will be included only if necessary.

only even expansions. Thus,

$$U' = \epsilon^2 U^{(2)} + \epsilon^4 U^{(4)} + \ldots. \tag{3.7}$$

$$V' = \epsilon^2 V^{(2)} + \epsilon^4 V^{(4)} + \ldots. \tag{3.8}$$

When these expansions are substituted in Equation (2.6) and other relevant equations, the requirement that each coefficient in the power series vanish individually leads to a set of linear differential equations with associated boundary conditions. These equations can be solved in sequence.

For ϵ^1 the differential equation is

$$L_1(W^1) \equiv D\Delta\Delta W^1 - h\lambda_o t^o_{ij} W^1_{,ij} = 0 \tag{3.9}$$

and the boundary conditions are

$$B_1(W^1) = 0 \tag{3.10}$$

$$B_2(W^1) = 0 \tag{3.11}$$

This is the linear eigenvalue problem for the buckling of a rectangular plate subject to edge compressions or displacements. It is now assumed that the edge displacements are such as to induce a hydrostatic plane stress,* that is,

$$\begin{aligned} U_E &= -\frac{1-\nu}{E}\,a \\ V_E &= -\frac{1-\nu}{E}\,b \\ t^o_{ij} &= -\,\delta_{ij} \end{aligned} \tag{3.12}$$

There exist an infinite number of eigenvalues and eigenfunctions. The normalized deflection function

$$W^1 = h \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{3.13}$$

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* This corresponds to the case of uniform heating of a thermally isotropic plate.

in which m and n are integers, automatically satisfies the boundary conditions. This fixes the physical meaning of ϵ as representing the amplitude of the first term in the perturbation expansion. The associated eigenvalue takes the familiar form

$$\lambda_o = \frac{D}{h}\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) \tag{3.14}$$

Any combination of m and n in the above expression can be identified as an eigenvalue of the differential equation. If only the first buckling mode is of interest, the lowest eigenvalue associated with the first buckling mode is obtained by choosing $m = n = 1$ regardless of the aspect ratio $\frac{a}{b}$ of the plate.

For ϵ^3 the differential equation is

$$L_1(W^3) = h\lambda_2 t^o_{ij} W^1_{,ij} + hT^2_{ij} W^1_{,ij} \tag{3.15}$$

in which

$$T^2_{ij} = \frac{1}{2}\left\langle W^1_{,i} W^1_{,j}\right\rangle \tag{3.16}$$

with W^1 given by Equation (3.13).

The associated boundary conditions are

$$B_1(W^3) = 0 \tag{3.17}$$

$$B_2(W^3) = 0 \tag{3.18}$$

The differential Equation (3.15) here is nonhomogeneous, but the associated homogeneous equation is identical with Equation (3.9). This homogeneous system has the nontrivial solution W^1. The nonhomogeneous differential equation therefore has a solution if and only if the right hand side of Equation (3.15) is orthogonal to W^1.[24] When Equation (3.9) is multiplied by W^3 and Equation (3.15) by W^1, after integration by parts and in view of the boundary conditions, this orthogonality condition becomes

$$\lambda_2 = - \frac{\int T^2_{ij} W^1_{,i} W^1_{,j} dA}{\int t^o_{ij} W^1_{,i} W^1_{,j} dA} \tag{3.19}$$

Furthermore, the solution is not unique. Any arbitrary multiple of W^1 added to a particular solution is also a solution of the differential Equation (3.15). Let $\overline{W}^3$ be a particular solution. Then W^3 is, in general, given by

$$W^3 = \overline{W}^3 + \alpha_3 W^1 \tag{3.20}$$

The choice of the value of α_3 is arbitrary. For convenience of computation let

$$\int t^o_{ij} W^1_{,i} W^3_{,j} = 0 \tag{3.21}$$

then

$$\alpha_3 = - \frac{\int t^o_{ij} W^1_{,i} \overline{W}^3_{,j} dA}{\int t^o_{ij} W^1_{,i} W^1_{,j} dA} \tag{3.22}$$

This is always possible, since in the present case

$$\int t^o_{ij} W^1_{,i} W^1_{,j} dA = - \int W^1_{,i} W^1_{,i} dA < 0 \tag{3.23}$$

Let the vector $\vec{T}$ denote any stress field T_{ij} symbolically, and let the inner product of two vectors $\vec{T}^\alpha$ and $\vec{T}^\beta$ be defined by

$$\vec{T}^\alpha \cdot \vec{T}^\beta = h \int T^\alpha_{ij} E^\beta_{ij} dA \tag{3.24}$$

in which E^β_{ij} is the strain associated with the stress T^β_{ij}. In view of the positive definiteness of the strain energy and of the symmetry of the stress-strain coefficients, it follows that $\vec{T}^\alpha \cdot \vec{T}^\alpha$ is positive definite and that $\vec{T}^\alpha \cdot \vec{T}^\beta = \vec{T}^\beta \cdot \vec{T}^\alpha$.

By Equation (3.16), through the application of Green's theorem, and in view of the boundary conditions, it can be shown that

$$\vec{T}^1 \cdot \vec{T}^2 = \frac{1}{2} h \int T^1_{ij} W^1_{,i} W^1_{,j} dA \tag{3.25}$$

Likewise we define

$$\vec{T} \cdot (W^m W^n) = h \int T_{ij} W^m_{,i} W^n_{,j} dA \tag{3.26}$$

Equation (3.19) can be written

$$\lambda_2 = - \frac{\vec{T}^2 \cdot \vec{T}^2}{\vec{T}^o \cdot \vec{T}^2} \tag{3.27}$$

Since, for positive λ, $\vec{T}^o \cdot \vec{T}^2$ is negative, [see (3.23)], and since $\vec{T}^2 \cdot \vec{T}^2$ is positive definite, λ_2 is always positive. This, in turn, confirms the well-known fact that the load parameter increases with increasing buckling amplitudes near the buckling point; the latter therefore represents a point of stable equilibrium.

For ϵ^5 the governing differential equation is

$$L_1(W^5) = h\lambda_2 t^o_{ij} W^3_{,ij} + h\lambda_4 t^o_{ij} W^1_{,ij} + hT^2_{ij} W^3_{,ij} + hT^4_{ij} W^1_{,ij} \tag{3.28}$$

in which

$$T^4_{ij} = \frac{1}{2} \left\langle W^1_{,i} W^3_{,j} + W^3_{,i} W^1_{,j} \right\rangle \tag{3.29}$$

with associated boundary conditions

$$B_1(W^5) = 0 \tag{3.30}$$

$$B_2(W^5) = 0 \tag{3.31}$$

As before, the right hand side of Equation (3.28) must satisfy the orthogonality condition if the equations has a solution for W^5. Thus,

$$\lambda_4 = - \frac{\int T^2_{ij} W^3_{,i} W^1_{,j} dA + \int T^4_{ij} W^1_{,i} W^1_{,j} dA}{\int t^o_{ij} W^1_{,i} W^1_{,j} dA} \tag{3.32}$$

Let $\overline{W}^5$ be a particular solution of Equation (3.28), then

$$W^5 = \overline{W}^5 + \alpha_5 W^1 \tag{3.33}$$

Let, for convenience,

$$\int t^o_{ij} W^1_{,i} W^5_{,j} dA = 0 \tag{3.34}$$

from which

$$\alpha_5 = - \frac{\int t^o_{ij} W^1_{,i} \overline{W}^5_{,j} dA}{\int t^o_{ij} W^1_{,i} W^1_{,j} dA} \tag{3.35}$$

In terms of the inner product notation, Equation (3.32) reduces to

$$\lambda_4 = - \frac{3\vec{T}^4 \cdot \vec{T}^2}{2\vec{T}^o \cdot \vec{T}^2} \tag{3.36}$$

in which Equation (3.29) has been utilized. Since $\vec{T}^4 \cdot \vec{T}^2$ can be either positive or negative, no conclusion can be drawn relative to the sign of the value of λ_4 .

The equations which contain higher powers of ϵ can be solved in the same manner; however, the calculations become exceedingly cumbersome. For the range of values considered here no further expansion has been found necessary.

We now consider the vibratory motion of the plate. It is noted that the method of solution in the dynamic case is similar to the one used above and hence only the essential points are presented.

The equation governing the motion of the plate is Equation (2.10). For the sake of convenience, it is presented again:

$$D\Delta\Delta W - h(\lambda t^o_{ij} + T'_{ij})\, w_{,ij} - ht'_{ij} W_{,ij} - \mu h w = 0 \tag{2.10}$$

with

$$t'_{ij} = \frac{1}{2} \left\langle W_{,i} w_{,j} + w_{,i} W_{,j} \right\rangle \tag{2.12}$$

subject to the boundary conditions

$$B_1(w) = 0 \tag{2.17}$$

$$B_2(w) = 0 \tag{2.18}$$

Here, λ, t^o_{ij}, T'_{ij} and W are now assumed to be known. The differential equations and the boundary conditions are linearly homogeneous in w, and once again we have an eigenvalue problem in which μ represents the eigenvalue. For each eigenvalue ${}_{pq}\mu$, there exists an eigenfunction ${}_{pq}w(x,y)$ which satisfies the differential equation as well as the boundary conditions. The prescripts p,q denote the pq^{th} mode of vibration.

We assume that the eigenfunction ${}_{pq}w$ and the associated eigenvalue ${}_{pq}\mu$ can be expanded in a power series in terms of the same parameter ϵ as in the static case, that is,

$$ {}_{pq}w = {}_{pq}w^{(0)} + \epsilon^2\, {}_{pq}w^{(2)} + \epsilon^4\, {}_{pq}w^{(4)} + \ldots . \tag{3.37}$$

$$ {}_{pq}\mu = {}_{pq}\mu^{(0)} + \epsilon^2\, {}_{pq}\mu^{(2)} + \epsilon^4\, {}_{pq}\mu^{(4)} + \ldots . \tag{3.38}$$

The fact that ${}_{pq}w$ and ${}_{pq}\mu$ are even expansions in ϵ may be easily verified upon substitution in the relevant equations. For the sake of brevity these steps are omitted here. In view of Equation (2.12) ${}_{pq}t'_{ij}$ can be expressed as

$$ {}_{pq}t'_{ij} = \epsilon\, {}_{pq}t^{(1)}_{ij} + \epsilon^3\, {}_{pq}t^{(3)}_{ij} + \epsilon^5\, {}_{pq}t^{(5)}_{ij} + \ldots . \tag{3.39}$$

with

$$ {}_{pq}t^{(n)}_{ij} = \frac{1}{2} \left\langle \sum_{s=1,3}^{n} (W^{(s)}_{,i}\, {}_{pq}w^{(n-s)}_{,j} + {}_{pq}w^{(s)}_{,i}\, W^{(n-s)}_{,j}) \right\rangle $$

Upon substitution of these perturbation expansions in Equation (2.10) a new sequence of differential equations is obtained whose solution follows procedures analogous to those presented for the static case.

From here on the prescripts p,q will be omitted, it being understood that $w^{(n)}$, $\mu^{(n)}$ and $t_{ij}^{(n)}$ denote the nth perturbation coefficients of the deflection, frequency squared and membrane stresses functions, respectively, for the pq^{th} mode of vibration of the plate. Whenever there is a possibility of confusion, or a specific mode of vibration is referred to, the prescripts will be added.

For ϵ^o the differential equation is

$$L_2(w^o) \equiv D\Delta\Delta w^o - h\lambda_o t_{ij}^o w^o_{,ij} - \mu^o h w^o = 0 \tag{3.40}$$

with the associated boundary conditions

$$B_1(w^o) = 0 \tag{3.41}$$

$$B_2(w^o) = 0 \tag{3.42}$$

This is satisfied by the normalized* function

$$w^o = \sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b} \tag{3.43}$$

in which p and q are integers, provided that

$$\mu^o = \left(\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right)\left[\frac{D}{h}\left(\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right) - \lambda_o\right] \tag{3.44}$$

The membrane stresses t^1_{ij} can now be obtained from

$$t^1_{ij} = \frac{1}{2}\left\langle W^1_{,i}w^o_{,j} + w^o_{,i}W^1_{,j}\right\rangle \tag{3.45}$$

---

* Naturally the linearized vibration solution is subject to an arbitrary amplitude factor.

For ϵ^2 the differential equation is

$$L_2(w^2) = h(\lambda_2 t^o_{ij} + T^2_{ij})w^o_{,ij} + ht^1_{ij}W^1_{,ij} + \mu^2 hw^o \tag{3.46}$$

and the boundary conditions are

$$B_1(w^2) = 0 \tag{3.47}$$

$$B_2(w^2) = 0 \tag{3.48}$$

As before, the right hand side of Equation (3.46) must satisfy an orthogonality condition* if a solution is to exist. After some manipulation this leads to

$$\mu^2 = \frac{(\lambda_2 \vec{T}^o + \vec{T}^2) \cdot (w^o w^o) + \vec{t}^1 \cdot \vec{t}^1}{h \int (w^o)^2 dA} \tag{3.49}$$

The solution of the differential Equation (3.46) is not unique; any multiple of w^o added to the particular solution is also a solution of the differential equation. For the sake of convenience we let

$$\int w^2 w^o dA = 0 \tag{3.50}$$

Thus, w^2 is determined and t^3_{ij} can now be obtained from

$$t^3_{ij} = \frac{1}{2} \left\langle W^1_{,i}w^2_{,j} + w^2_{,i}W^1_{,j} + W^3_{,i}w^o_{,j} + w^o_{,i}W^3_{,j} \right\rangle \tag{3.51}$$

For ϵ^4 the differential equation is

$$\begin{aligned} L_2(w^4) = {} & ht^o_{ij}(\lambda_4 w^o_{,ij} + \lambda_2 w^2_{,ij}) + hT^4_{ij}w^o_{,ij} + hT^2_{ij}w^2_{,ij} \\ & + ht^3_{ij}W^1_{,ij} + ht^1_{ij}W^3_{,ij} + \mu^4 hw^o + \mu^2 hw^2 \end{aligned} \tag{3.52}$$

---

* Note that this orthogonality condition is different from the one pertaining to the static case.

and the associated boundary conditions are

$$B_1(w^4) = 0 \tag{3.53}$$

$$B_2(w^4) = 0 \tag{3.54}$$

Again, the orthogonality condition determines the value of

$$\mu^4 = \frac{(\lambda_4\vec{T}^0 + \vec{T}^4)\cdot(w^0w^0) + (\lambda_2\vec{T}^0 + \vec{T}^2)\cdot(w^2w^0) + 2\vec{t}^3 \cdot \vec{t}^1 - \vec{t}^1 \cdot (w^1w^2)}{h \int (w^0)^2 dA} \tag{3.55}$$

while the deflection function w^4 satisfies

$$\int w^4 w^0 dA = 0 \tag{3.56}$$

Since the static deflection is truncated at the coefficient W^5, there is no sense in pursuing the solution of the dynamic problem beyond this point.

The results of these calculationsare given in Appendix A for the general case of a rectangular plate. The first part deals with the static problem. Algebraic expressions are given for the expansion terms in the deflection $W(x,y)$, the additional stresses $T'_{ij}(x,y)$, the load parameter λ, and the additional membrane displacements $U'(x,y)$ and $V'(x,y)$. These are not necessarily based on the assumption that the plate buckles freely immediately after its unbuckled equilibrium configuration becomes unstable; however, the case of $m = n = 1$ is the only one which has practical significance.

The dynamic response for the same case is computed next. Again general algebraic expressions are given for the vibration modes, membrane stresses, and frequency parameters. Only the lowest two modes $p = 1$, $q = 1$ and $p = 2$, $q = 1$ are considered; an obvious, and trivial, extension is easily obtained for $p = 1, q = 2$ through a suitable exchange of variables.

Higher buckling modes (say, $m = 2, n = 1$) are of course associated with larger critical buckling parameters; however, as the lowest buckling parameter λ_o is exceeded, at least one frequency becomes imaginary and the associated unbuckled equilibrium configuration becomes unstable and hence physically meaningless. Nevertheless it is conceivable that if the plate were forced into one of these higher buckling modes (perhaps through the application of kinematic constraints), its equilibrium may again become stable for sufficiently large buckling amplitude. A necessary and sufficient criterion for such a condition is that the square of the smallest frequency of vibration becomes again positive.

This may be physically significant. As has been observed and commented on repeatedly, buckled plates often snap from their original buckling configuration into another one. Just when this type of "secondary buckling" takes place is conjectural and has been the object of some speculation; for example it has been postulated that a suitable criterion is obtained when the energies in the primary and secondary states are equated.[25] In any event it is safe to state that a snap-through from a stable configuration into an unstable one can be ruled out. The lowest loading parameter at which the secondary configuration becomes stable may therefore be considered a lower bound to the secondary buckling parameter.

Charts showing the frequencies of vibration for specific cases are given in the present paper for several such higher buckling modes, and in Appendix A are included the general algebraic expressions for the vibration modes, membrane stresses and frequency parameters of the lowest two modes $p = 1, q = 1$ and $p = 2, q = 1$ if the plate has buckled into the second mode $m = 2, n = 1$. The general algebraic expressions of μ^o and μ^2 are given for any vibration mode and for any buckling mode. These expressions are generally rather complex and have therefore been deleted from the main body of the paper.

For the special (and, near the buckling point, most important) condition of $m = n = p = q = 1$, the formulas become much simpler. Since, for that case, $w^o = W^1$, it follows from Equation (3.40) that

$$\mu^o = 0 \tag{3.57}$$

as expected, which in turn implies that $\vec{t}^1 = 2\vec{T}^2$ and $w^2 = 3W^3$. When these relations are substituted in Equations (3.49) and (3.55), it follows, after some manipulation, that

$$\mu^2 = \frac{4\vec{T}^2 \cdot \vec{T}^2}{h \int (W^1)^2 dA} = 2(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}) \lambda_2 \tag{3.58}$$

$$\mu^4 = \frac{2\lambda_4\vec{T}^o \cdot \vec{T}^2 + 3\lambda_2\vec{T}^o \cdot (W^3W^1) + 15\vec{T}^4 \cdot \vec{T}^2}{h \int (W^1)^2 dA} = 4(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}) \lambda_4 \tag{3.59}$$

For the rate of change of frequency of vibration with respect to the load parameter in the neighborhood of buckling ($\epsilon = 0$) one obtains

$$\lim_{\epsilon \to 0} \frac{d\mu}{d\lambda} = \lim_{\epsilon \to 0} \frac{d\mu}{d\epsilon} \Big/ \frac{d\lambda}{d\epsilon} = \frac{\mu^{(2)}}{\lambda_2} \tag{3.60}$$

For the vibration mode $p = q = 1$, $\mu^{(2)}$ is positive and so is the value of $\frac{d\mu}{d\lambda}$ at the point of buckling, as anticipated since the plate is stable in the immediate post-buckling neighborhood. A similar, though less important, conclusion is reached for any vibration mode satisfying $p=m$, $q=n$.

CHAPTER IV

THE ENERGY METHOD SOLUTION

In the perturbation method, the rapidity of convergence of the perturbation series is always an issue. In some problems the series converges fairly rapidly, in others it converges only for a rather small range of values of the load parameter λ. An indication (though not fully conclusive) of the convergence of the perturbation series is the agreement between the results obtained from the truncation at the term ϵ^n and ϵ^{n-1}. The present calculations show satisfactory convergence for a technically significant range of the load parameter. Nevertheless the truncated expressions become unreliable, as expected, when the buckling amplitudes reach very large values. To cover this range at least approximately an energy method is employed in this chapter.

The static condition is analyzed by a method similar to the one of Marguerre-Papkovitch.[9] The deflection of the plate is assumed to be expressed by means of

$$W = C_1 W^1 + C_2 W^2 + C_3 W^3 \tag{4.1}$$

in which W^1, W^2 and W^3 are geometrically admissible functions and C_1, C_2 and C_3 are parameters whose values are to be determined from the theorem of minimum potential energy. With this assumed deflection function, the additional membrane stresses T'_{ij} can be obtained from Equation (2.8), that is,

$$T'_{ij} = C_1^2 T_{ij}^{11} + C_2^2 T_{ij}^{22} + C_3^2 T_{ij}^{33} + C_1 C_2 T_{ij}^{12} + C_1 C_3 T_{ij}^{13} + C_2 C_3 T_{ij}^{23} \tag{4.2}$$

in which

$$T^{11}_{ij} = \frac{1}{2} \left\langle W^1_{,i} W^1_{,j} \right\rangle$$

$$T^{22}_{ij} = \frac{1}{2} \left\langle W^2_{,i} W^2_{,j} \right\rangle$$

$$T^{33}_{ij} = \frac{1}{2} \left\langle W^3_{,i} W^3_{,j} \right\rangle$$

$$T^{12}_{ij} = \frac{1}{2} \left\langle W^1_{,i} W^2_{,j} + W^2_{,i} W^1_{,j} \right\rangle$$

$$T^{13}_{ij} = \frac{1}{2} \left\langle W^1_{,i} W^3_{,j} + W^3_{,i} W^1_{,j} \right\rangle$$

$$T^{23}_{ij} = \frac{1}{2} \left\langle W^2_{,i} W^3_{,j} + W^3_{,i} W^2_{,j} \right\rangle$$

The additional potential energy V, that is the difference in the potential energies of the buckled and unbuckled states, is defined by

$$V = U_b + U_m - \lambda W_e \tag{4.3}$$

in which

$$U_b \equiv \frac{D}{2} \int [(1-\nu) W_{,ij} W_{,ij} + \nu W_{,ii} W_{,jj}] dA$$

$$U_m \equiv \frac{h}{2} \int T'_{ij} E'_{ij} dA = \frac{1}{2} \vec{T}' \cdot \vec{T}'$$

$$W_e \equiv - \frac{h}{2} \int t^o_{ij} W_{,i} W_{,j} dA$$

After application of Green's theorem, membrane stress equilibrium equations ($T'_{ij,j} = 0$) and boundary conditions, the membrane strain energy U_m is now given by

$$U_m = \frac{h}{4} \int T'_{ij} W_{,i} W_{,j} dA \tag{4.4}$$

If the edges of the plate are simply supported, the bending strain energy U_b reduces to

$$U_b = \frac{D}{2} \int W_{,ii} W_{,jj} dA \tag{4.5}$$

Since $t^o_{ij} = -\delta_{ij}$, W_e becomes

$$W_e = \frac{h}{2} \int W_{,i}W_{,i}dA \tag{4.6}$$

Upon substitution of Equations (4.1) and (4.2) into Equation (4.3) we have

$$\begin{aligned} V = {} & (U_b^{11} - \lambda W_e^{11})C_1^2 + (U_b^{22} - \lambda W_e^{22})C_2^2 + (U_b^{33} - \lambda W_e^{33})C_3^2 \\ & + \frac{1}{2}(C_1^2\vec{T}^{11} + C_2^2\vec{T}^{22} + C_3^2\vec{T}^{33} + C_1C_2\vec{T}^{12} + C_1C_3\vec{T}^{13} + C_2C_3\vec{T}^{23}) \qquad (4.7) \\ & \cdot (C_1^2\vec{T}^{11} + C_2^2\vec{T}^{22} + C_3^2\vec{T}^{33} + C_1C_2\vec{T}^{12} + C_1C_3\vec{T}^{13} + C_2C_3\vec{T}^{23}) \end{aligned}$$

in which*

$$U_b^{11} \equiv \frac{D}{2} \int W^1_{,ii}W^1_{,jj}dA$$

$$U_b^{22} \equiv \frac{D}{2} \int W^2_{,ii}W^2_{,jj}dA$$

$$U_b^{33} \equiv \frac{D}{2} \int W^3_{,ii}W^3_{,jj}dA$$

$$W_e^{11} \equiv \frac{h}{2} \int W^1_{,i}W^1_{,i}dA$$

$$W_e^{22} \equiv \frac{h}{2} \int W^2_{,i}W^2_{,i}dA$$

$$W_e^{33} \equiv \frac{h}{2} \int W^3_{,i}W^3_{,i}dA$$

Setting the first variation of the potential energy equal to zero leads

* Terms such as U_b^{12}, W_e^{12} etc. may also appear; however, if $W^{(n)}$ are chosen to be orthogonal functions, these terms vanish from the above expression.

to

$$
\begin{aligned}
&2(U_b^{11} - \lambda W_e^{11})C_1 + 2(\vec{T}^{11}\cdot\vec{T}^{11})C_1^3 + (\vec{T}^{12}\cdot\vec{T}^{12} + 2\vec{T}^{11}\cdot\vec{T}^{22})C_1C_2^2 \\
&+ (\vec{T}^{13}\cdot\vec{T}^{13} + 2\vec{T}^{11}\cdot\vec{T}^{33})C_1C_3^2 + 3(\vec{T}^{11}\cdot\vec{T}^{12})C_1^2C_2 + 3(\vec{T}^{11}\cdot\vec{T}^{13})C_1^2C_3 \\
&+ 2(\vec{T}^{11}\cdot\vec{T}^{23} + \vec{T}^{12}\cdot\vec{T}^{13})C_1C_2C_3 + (\vec{T}^{22}\cdot\vec{T}^{12})C_2^3 + (\vec{T}^{22}\cdot\vec{T}^{13} + \vec{T}^{12}\cdot\vec{T}^{23})C_2^2C_3 \\
&+ (\vec{T}^{33}\cdot\vec{T}^{12} + \vec{T}^{13}\cdot\vec{T}^{23})C_1C_3^2 + (\vec{T}^{33}\cdot\vec{T}^{13})C_3^3 = 0 \qquad (4.8a)
\end{aligned}
$$

$$
\begin{aligned}
&2(U_b^{22} - \lambda W_e^{22})C_2 + 2(\vec{T}^{22}\cdot\vec{T}^{22})C_2^3 + (\vec{T}^{12}\cdot\vec{T}^{12} + 2\vec{T}^{11}\cdot\vec{T}^{12})C_1^2C_2 \\
&+ (\vec{T}^{23}\cdot\vec{T}^{23} + 2\vec{T}^{22}\cdot\vec{T}^{33})C_2C_3^2 + 3(\vec{T}^{22}\cdot\vec{T}^{12})C_1C_2^2 + 3(\vec{T}^{22}\cdot\vec{T}^{23})C_2^2C_3 \\
&+ 2(\vec{T}^{22}\cdot\vec{T}^{13} + \vec{T}^{12}\cdot\vec{T}^{23})C_1C_2C_3 + (\vec{T}^{33}\cdot\vec{T}^{23})C_3^3 + (\vec{T}^{33}\cdot\vec{T}^{12} + \vec{T}^{13}\cdot\vec{T}^{23})C_1C_3^2 \\
&+ (\vec{T}^{11}\cdot\vec{T}^{23} + \vec{T}^{12}\cdot\vec{T}^{13})C_1^2C_3 + (\vec{T}^{11}\cdot\vec{T}^{12})C_1^3 = 0 \qquad (4.8b)
\end{aligned}
$$

$$
\begin{aligned}
&2(U^{33} - \lambda W^{33})C_3 + 2(\vec{T}^{33}\cdot\vec{T}^{33})C_3^3 + (\vec{T}^{13}\cdot\vec{T}^{13} + 2\vec{T}^{11}\cdot\vec{T}^{33})C_1^2C_3 \\
&+ (\vec{T}^{23}\cdot\vec{T}^{23} + 2\vec{T}^{22}\cdot\vec{T}^{33})C_2^2C_3 + 3(\vec{T}^{33}\cdot\vec{T}^{13})C_1C_3^2 + 3(\vec{T}^{33}\cdot\vec{T}^{23})C_2C_3^2 \\
&+ 2(\vec{T}^{33}\cdot\vec{T}^{12} + \vec{T}^{13}\cdot\vec{T}^{23})C_1C_2C_3 + (\vec{T}^{11}\cdot\vec{T}^{13})C_1^3 + (\vec{T}^{11}\cdot\vec{T}^{23} + \vec{T}^{12}\cdot\vec{T}^{13})C_1^2C_2 \\
&+ (\vec{T}^{22}\cdot\vec{T}^{13} + \vec{T}^{12}\cdot\vec{T}^{23})C_1C_2^2 + (\vec{T}^{22}\cdot\vec{T}^{23})C_2^3 = 0 \qquad (4.8c)
\end{aligned}
$$

C_1, C_2 and C_3 can be solved in terms of the load parameter λ from Equations (4.8).

The deflection functions of the lowest buckling mode can be assumed to be of the following form

$$
\begin{aligned}
W^1 &= \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y \\
W^2 &= \sin\frac{3\pi}{a}x \sin\frac{\pi}{b}y \qquad (4.9) \\
W^3 &= \sin\frac{\pi}{a}x \sin\frac{3\pi}{b}y
\end{aligned}
$$

This assumption is not without justification. W^1 is the buckling mode as predicted in the linear theory. The formation of a W^2 wave along the unloaded edges and in the direction of the loads has been observed experimentally in the buckling of a simply supported plate subjected to longitudinal edge compressions. It is therefore reasonable to include both W^2 and W^3 functions in the present problem. Note that W^1 is also the first term of the perturbation series for W and $(W^2 + W^3)$ is the second term of the perturbation series for a square plate.

An exact solution to the dynamic problem is generally out of the question, in spite of its linearity, because of the presence of functions of x and y as coefficients in the relevant differential equations. For this type of problem the Galerkin method (which, for conversative systems of the present kinds, represents essentially a modified energy method) yields comparatively good approximations which are known to constitute upper bounds to the exact eigenvalues.*

If the vibration mode is assumed to be of the form

$$w = \sum_{n=1}^{N} a_n w_n(x,y) \tag{4.10}$$

then this technique leads to the linear system

$$\sum_{n=1}^{N} a_n P_{mn} = 0 \quad (m = 1, 2, \ldots, N) \tag{4.11}$$

in which

$$P_{mn} = P_{nm} = D \int (\Delta w^n)(\Delta w^m) dA + \lambda \vec{T}^o \quad (w^n w^m) + \vec{T}' \cdot (w^n w^m) + \vec{t}^n \cdot (W\, w^m) - \mu h \int w^n w^m dA \tag{4.12}$$

with

$$t^n_{ij} \equiv \frac{1}{2} \left\langle W_{,i} w^n_{,j} + w^n_{,i} W_{,j} \right\rangle \tag{4.13}$$

---

* No such statement can be made here, of course, as long as the static problem itself has been solved only approximately.

These equations have a non-trivial solution for a_n if

$$\text{determinant} \quad |P_{mn}| = 0 \tag{4.14}$$

from which μ is computed.

In the present analysis four terms have been used in the approximating series, namely,

$$\begin{aligned} w^1 &= \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ w^2 &= \sin \frac{\pi x}{a} \sin \frac{3\pi y}{b} \\ w^3 &= \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \\ w^4 &= \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b} \end{aligned} \tag{4.15}$$

The squares of the frequencies of the various modes are plotted as functions of the load parameter λ, with the results shown in the chart.

To determine the stability and instability of the buckling modes it is necessary to examine the second variation of the potential energy V. The latter is given in Equation (4.3), which, for convenience, may be written symbolically

$$V = U_b(W,W) + U_m \left\langle WW \right\rangle \left\langle WW \right\rangle - \frac{1}{2} \lambda \vec{T}^o \cdot (WW) \tag{4.16}$$

The following expansions identities are also useful:

$$\begin{aligned} U_b(W+w) &= U_b(W,W) + 2U_b(W,w) + U_b(w,w) \\ U_m(W+w) &= U_m \left\langle WW \right\rangle \left\langle WW \right\rangle + 4U_m \left\langle WW \right\rangle \left\langle Ww \right\rangle \\ &\quad + 2U_m \left\langle WW \right\rangle \left\langle ww \right\rangle + 4U_m \left\langle Ww \right\rangle \left\langle Ww \right\rangle \\ &\quad + 4U_m \left\langle Ww \right\rangle \left\langle ww \right\rangle + U_m \left\langle ww \right\rangle \left\langle ww \right\rangle \\ \vec{T}^o \cdot (W+w) &= \vec{T}^o \cdot (WW) + 2\vec{T}^o \cdot (Ww) + \vec{T}^o \cdot (ww) \end{aligned} \tag{4.17}$$

A configuration is in equilibrium if the potential energy assumes a stationary value. By standard methods this leads to Equation (2.6) in the present problem. The second variation of the potential energy, which determines the stability or instability of the buckled state, takes the form

$$\delta^2 V(WW;\eta\eta) = U_b(\eta,\eta) + 2U_m \left\langle WW \right\rangle \left\langle \eta\eta \right\rangle + 4U_m \left\langle W\eta \right\rangle \left\langle W\eta \right\rangle - \frac{1}{2}\lambda \vec{T}^o \cdot (\eta\eta) \tag{4.18}$$

After some integrations by parts and upon application of the boundary conditions, this leads to

$$\delta^2 V(WW;\eta\eta) = \int \left[\frac{D}{2}\eta_{,iijj} - \frac{h}{2}(\lambda t^o_{ij} + T'_{ij})\eta_{,ij} - \frac{h}{2}\tau'_{ij}W_{,ij}\right]\eta \, dA \tag{4.19}$$

in which

$$\tau'_{ij} \equiv \frac{1}{2}\left\langle W_{,i}\eta_{,j} + \eta_{,i}W_{,j}\right\rangle \tag{4.20}$$

It may be of interest to note that in view of Equation (2.10) the eigenvalues μ_n are equal to the stationary values of this expression provided the function $\eta(x,y)$ is chosen to be the associated normalized vibration mode $w_n(x,y)$. Since positive values for all μ_n have previously been identified with stability this confirms the familiar connection between stability and the positive definiteness of the second variation of the potential energy.

It is recalled that λ is the ratio of the edge displacement to that required for the initial instability. Now let γ be the ratio of the edge compressive force caused by the prescribed edge displacement to that required for initial instability. Then γ is related to λ

by the equation

$$\gamma = \frac{\int (\lambda t_{ij}^{o} + T_{ij}^{\prime})ds}{\int \lambda_o t_{ij}^{o} \; ds} \tag{4.21}$$

in which the integrals are along a loaded edge. The buckled state is often characterized by its γ versus λ curve, i.e., the load-shortening curve. The intersection of the load-shortening curve of one mode (say the symmetric mode which corresponds to the lowest buckling load) with the load-shortening curve of another mode (the antisymmetric mode which corresponds to the next lowest buckling load) usually indicates a possibility of the change of buckling modes. Just when and where this type of secondary buckling takes place is conjectural. Various authors[25,26] consider it reasonable to apply the equal energy criterion to determine the change of buckling modes. Hence, the primary buckling mode may change to the secondary buckling mode when

$$V_1 = V_2 \tag{4.22}$$

in which V_1 is the potential energy associated with the primary buckling mode and V_2 that associated with the secondary buckling mode.

In the present analysis, the vibration method and equal energy criterion are used to determine the stability of the buckling modes and changes of buckling modes. In addition to the problem stated in Chapter II, the stability and change of buckling modes of a simply supported rectangular plate subjected to uniaxial edge compression is also investigated by the present method. The details of this analysis are presented in Appendix B.

CHAPTER V

RESULTS AND DISCUSSION

Charts showing the frequencies of vibration and the load-shortening curves are given in nondimensional quantities μ', λ' and λ'', in which $\mu' = \mu/4(\frac{\pi}{a})^4 \frac{D}{h} \rho$, $\lambda' = \lambda/2(\frac{\pi}{a})^2 \frac{D}{h}$ and $\lambda'' = \lambda/4(\frac{\pi}{a})^2 \frac{D}{h}$. All calculations are based upon the value of Poisson's ratio $\nu = .30$.

Figure 1a shows the relation between μ' and λ' for small values of λ' for a square plate subjected to plane hydrostatic pressure. The results are obtained from the perturbation series which converges satisfactorily for $\lambda' < 16$. Only the two lowest vibration modes, i.e., $p = 1$, $q = 1$ and $p = 2$, $q = 1$, about the lowest buckling configuration $(m = n = 1)$ are plotted. It is interesting to note that μ' increases practically linearly with λ' in the vicinity of initial instability for both the symmetric $(p = q = 1)$ and antisymmetric $(p = 2, q = 1)$ vibration modes.

The frequency of the symmetric vibration mode is strongly affected by the increase of λ', the rate of increase of μ' after buckling being twice as much as the rate of decrease before buckling. For example, with $\lambda' = 4$ and $\mu' = 5.8$, the "stiffness" of the plate has increased to 2.41 times that of the unbuckled state while the maximum deflection at the center of the plate is only 1.5 h. This rapid increase in the stiffness after buckling is important in flutter analysis. In general, the vibration mode associated with the initial buckling mode, that is, $p = m$ and $q = n$, is the mode affected most strongly by the increase of λ'. For further increase of λ' the frequency of the symmetric vibration mode becomes higher than that of the antisymmetric mode. This is not unreasonable since the antisymmetric vibration is primarily inextensional while the symmetric vibration is primarily extensional.

The results of the same problem as shown in Figure 1a but for a larger range of values of λ' are shown in Figure 1b. The solid lines represent the perturbation solution up to $\lambda' = 50$; however, the results become less reliable since, for $\lambda' > 20$, the perturbation series for the symmetric vibration mode converges rather poorly. In contrast, for the antisymmetric vibration mode it still converges satisfactorily for values of λ' up to 40. The dashed lines represent the results of Equation (4.14) when the approximating series takes the form of Equation (4.15). It is noted that after a further increase of λ', the frequency of the symmetric vibration mode increases less rapidly and eventually becomes again less than that of the antisymmetric vibration mode. This is due to the fact that for large values of λ' two nodal lines appear in the symmetric vibration mode, which therefore becomes more nearly inextensional. Figure 2 shows the shapes of the symmetric and antisymmetric vibration modes for various values of λ'.

Figures 3a and 3b show the relation between γ and λ' for the same case of a square plate subjected to plane hydrostatic pressure. The perturbation results are shown in Figure 3a, the energy method results in Figure 3b. The rate of increase of γ after buckling is only one fourth as much as that before buckling (as against one half in the case of uniaxial edge compression).

Figure 4 shows the relations between μ' and λ' for rectangular plates of various aspect ratios. The plates are assumed to be forced

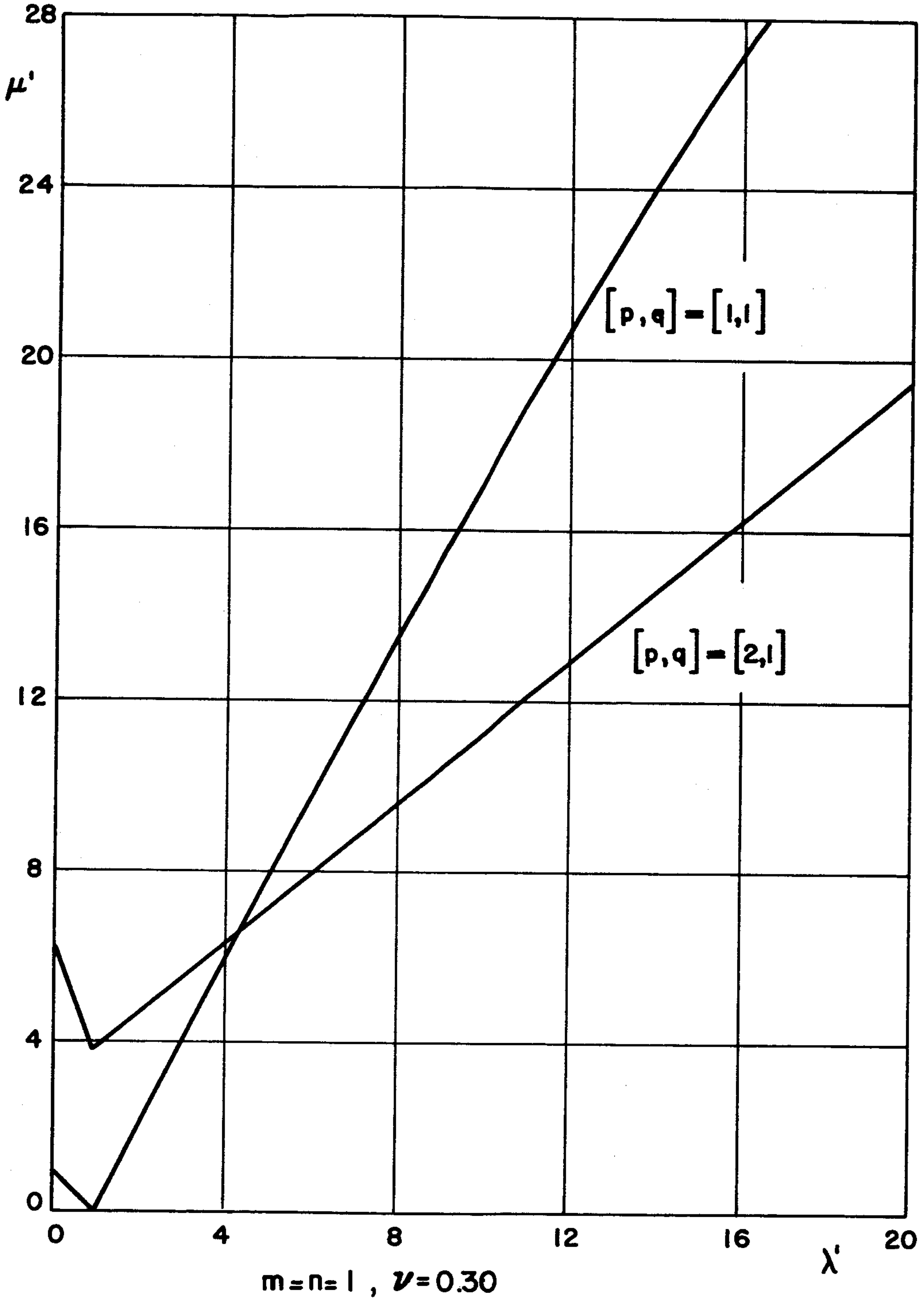

Figure 1a. Nondimensional Frequency Squared-Load Curves for Square Plate under Hydrostatic Pressure.

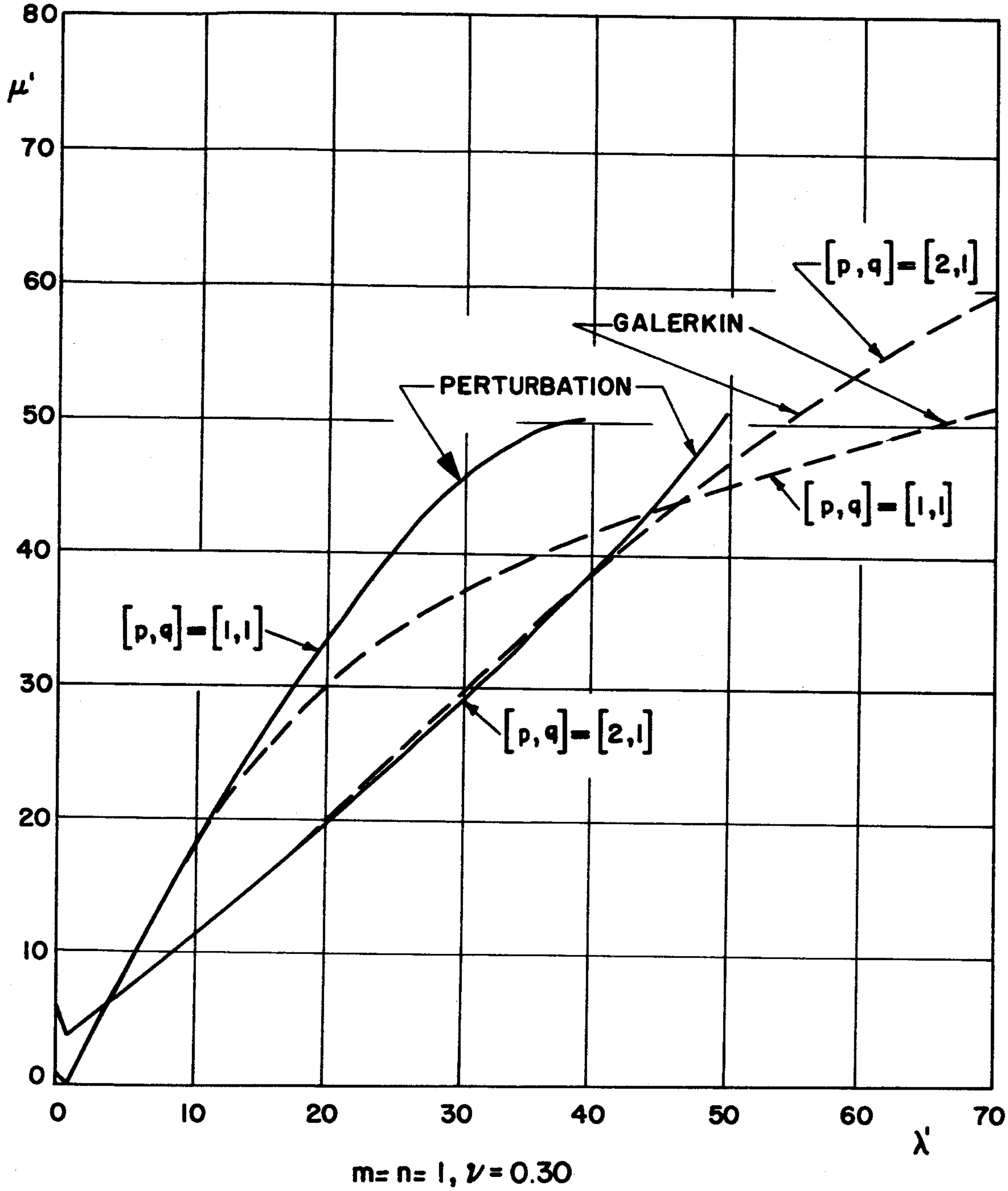

Figure 1b. Nondimensional Frequency Squared-Load Curves for Square Plate under Hydrostatic Pressure (same as Figure 1a but extended range).

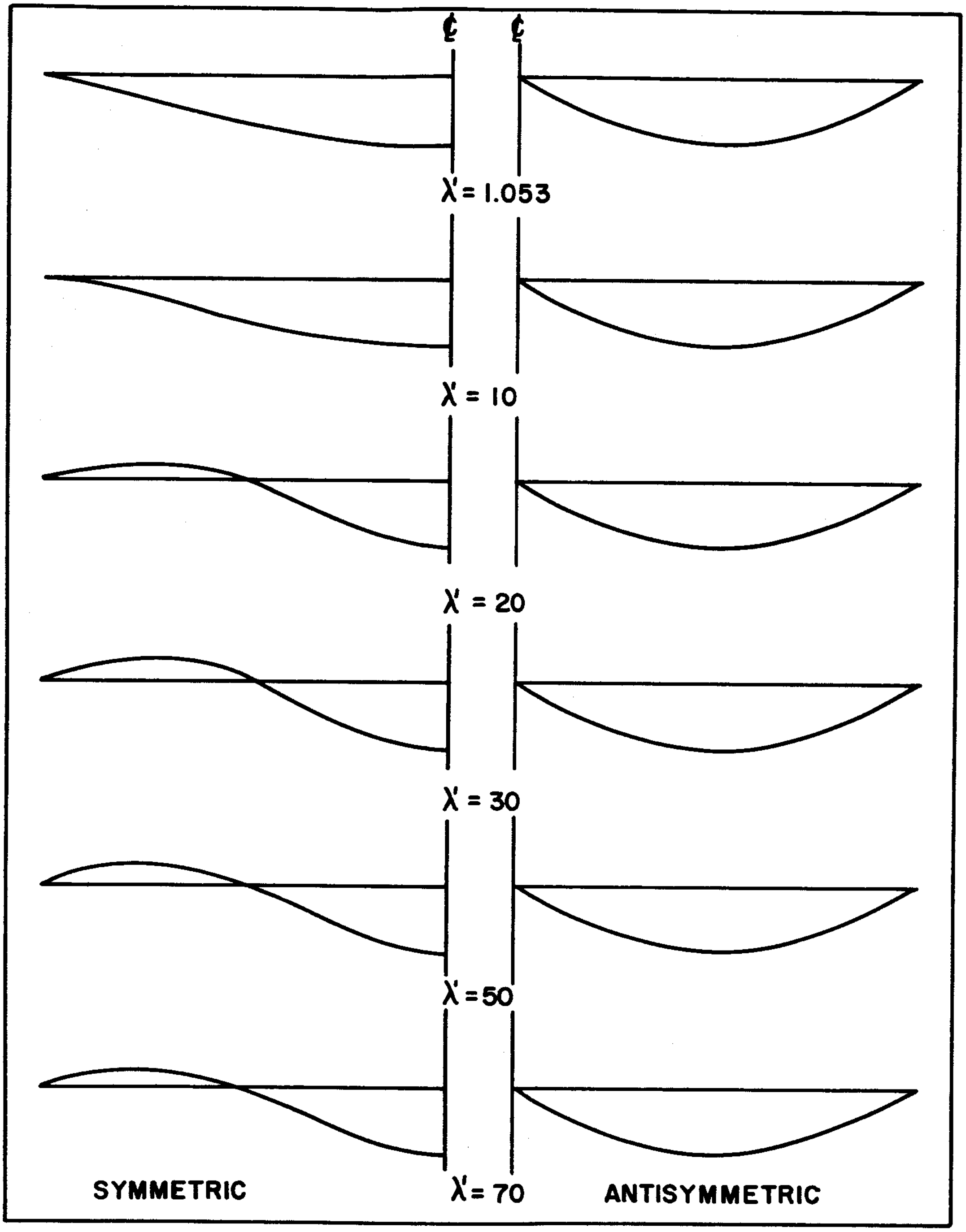

Figure 2. Modes of Vibration for Square Plate under Hydrostatic Pressure (only half plate is shown).

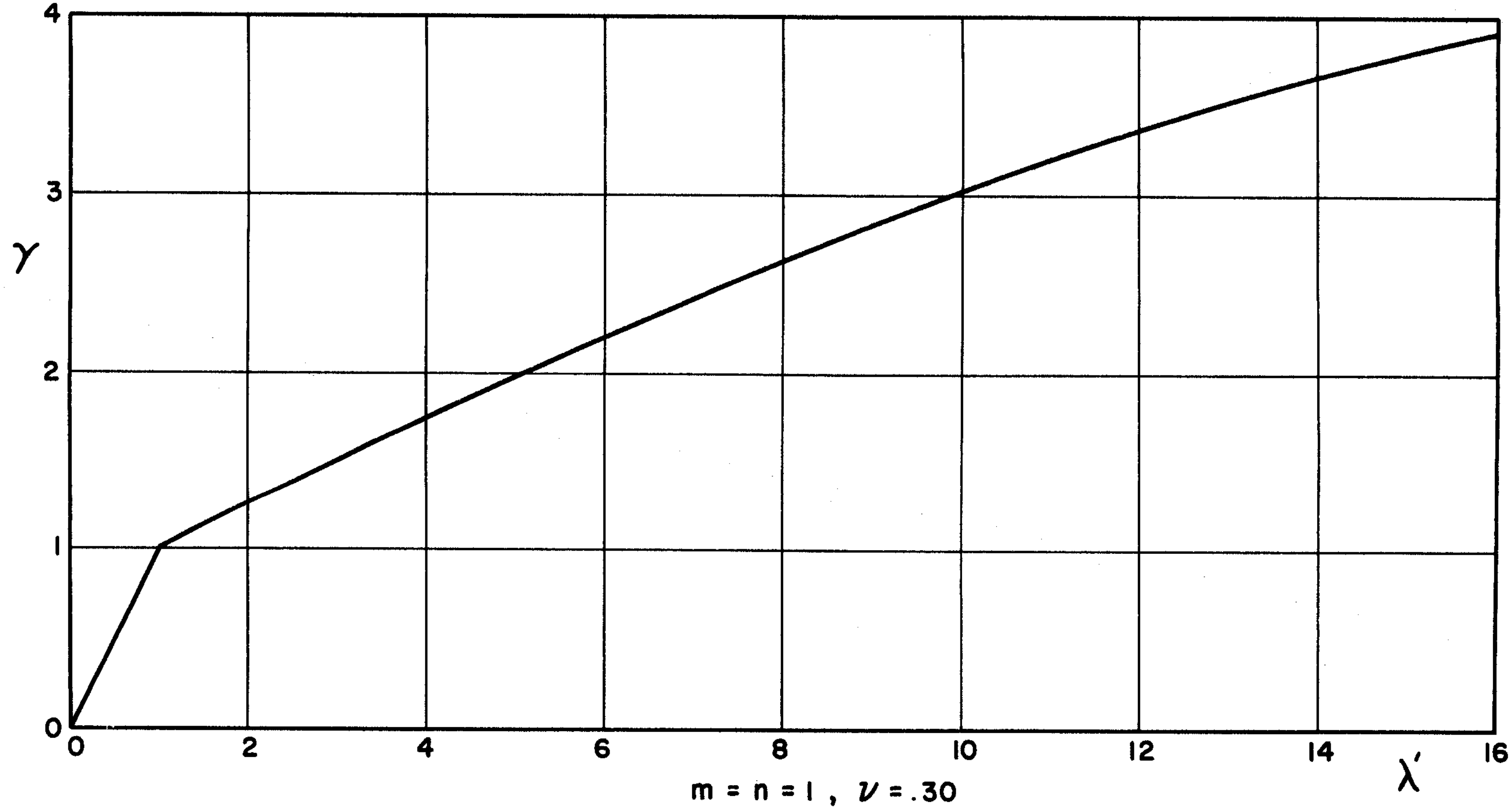

Figure 3a. Nondimensional Load-Shortening Curve for Square Plate under Hydrostatic Pressure.

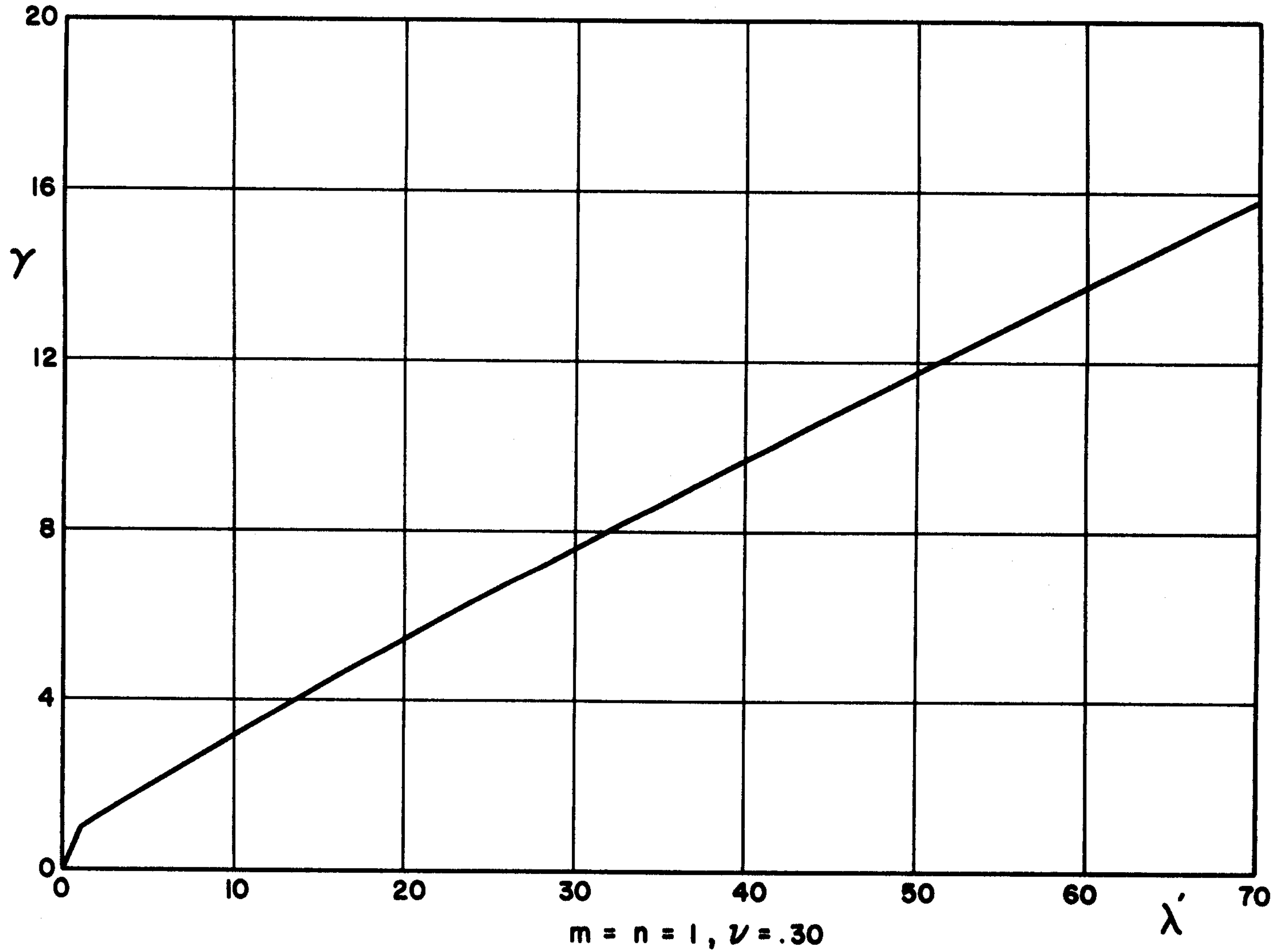

Figure 3b. Nondimensional Load-Shortening Curve for Square Plate under Hydrostatic Pressure (same as Figure 3a but extended range).

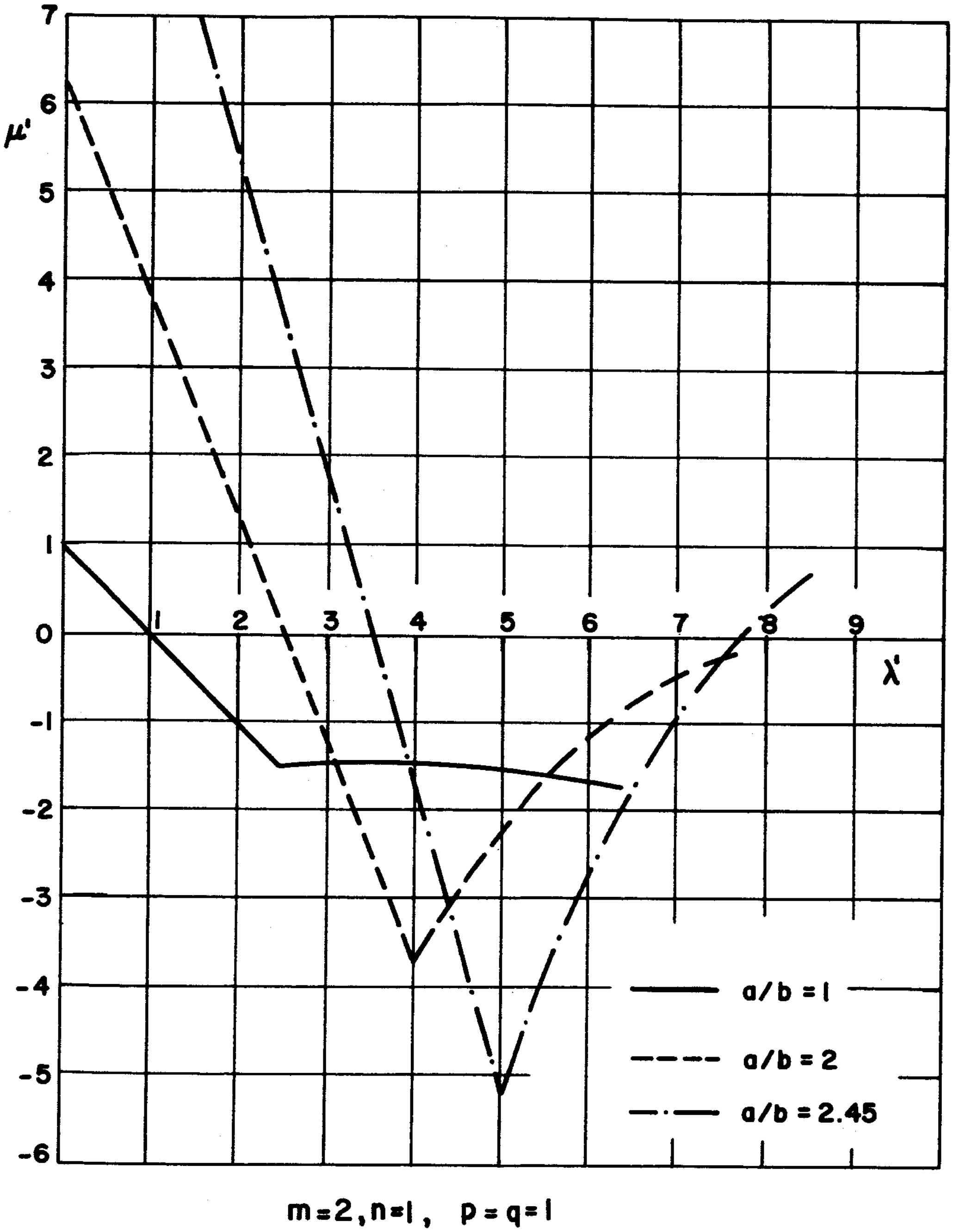

Figure 4. Nondimensional Frequency Squared-Load Curves for Rectangular Plate under Hydrostatic Pressure Second Buckling Mode.

into the second buckling mode ($m = 2$ and $n = 1$) through the application of artificial kinematic constraints which, however, do not restrict the freedom of dynamic vibratory motion. For $m = 2$, $n = 1$, the vibration mode associated with negative μ' corresponds to $p = q = 1$. For all other modes, μ' is positive in the vicinity of the initial instability and up to values of λ' which are of interest to us; hence they are not considered here.

The solid, dashed and dashed-dotted curves represent μ' versus λ' for the $p = q = 1$ vibration mode of plates of aspect ratio 1, 2 and 2.45, respectively. It is noted that μ' remains negative for all values of $\lambda' > 1$ for aspect ratios of 1 and 2, respectively, at least within the limit of the truncated series.* For an aspect ratio of 2.45 μ' becomes positive at $\lambda' = 7.70$; moreover, the truncated series shows satisfactory convergence for the range of values considered. This means that the $m = 2$, $n = 1$ buckling configuration will become stable even after the removal of the artificial kinematic constraints for sufficiently large values of λ'. In this case secondary buckling from the fundamental mode into the second mode may occur; in contrast such secondary buckling is ruled out for a square plate under hydrostatic pressure.

Figures 5, 6 and 7 show the load-shortening curves of the lowest buckling configurations ($m = 1$, $n = 1$ and $m = 2$, $n = 1$) for the plates considered in Figure 4. For a square plate (Figure 5) the edge displacement in the antisymmetric buckling configuration ($m = 2$, $n = 1$) increases with decreasing load when $\lambda' > 7.0$, confirming the previous

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* Slow convergence raises doubts as to the reliability of this statement for $a/b = 2$.

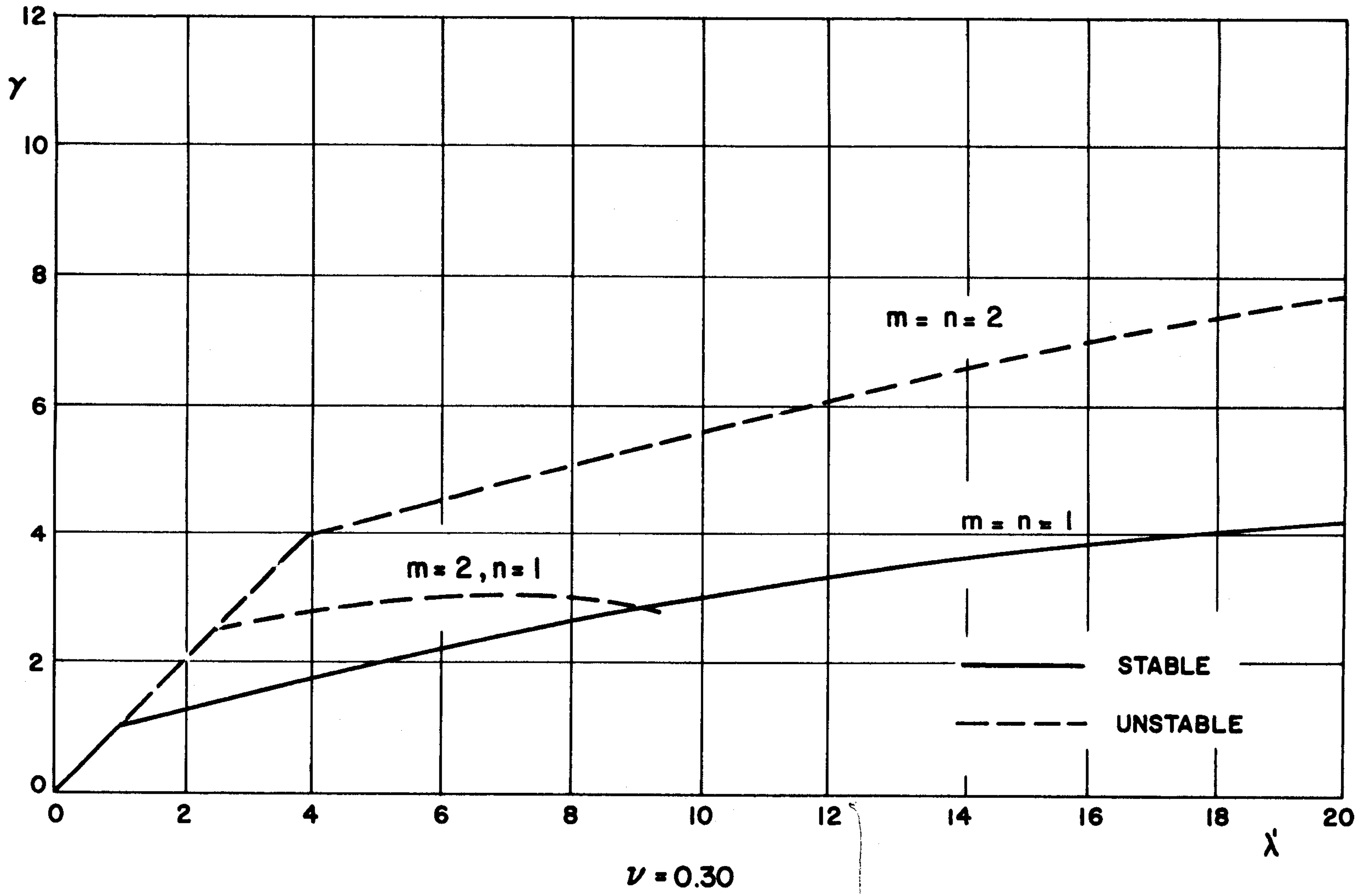


Figure 5. Nondimensional Frequency Squared-Load Curves for Square Plate under Hydrostatic Pressure.

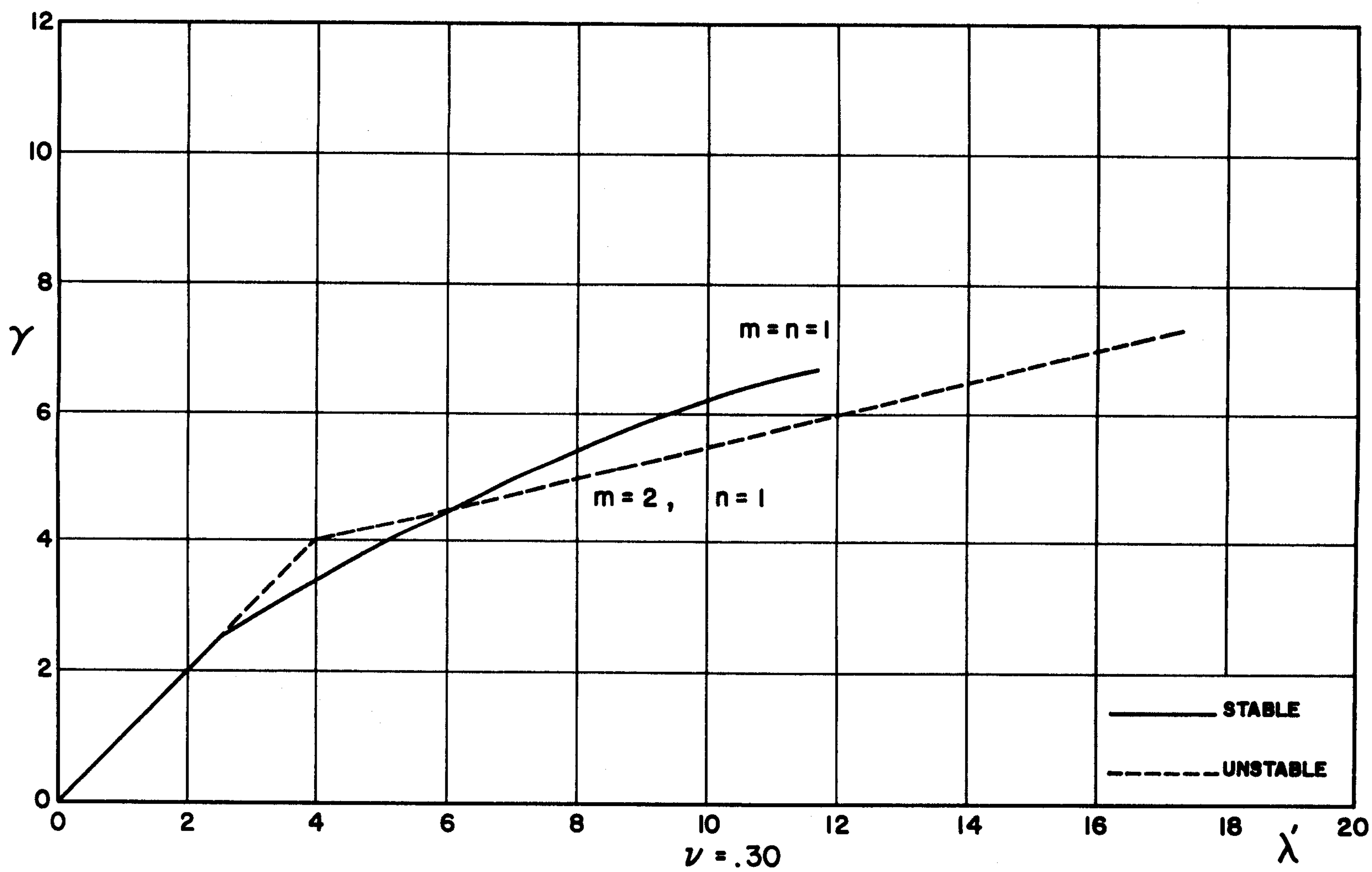


Figure 6. Nondimensional Load-Shortening Curves for Rectangular Plate under Hydrostatic Pressure a/b = 2.

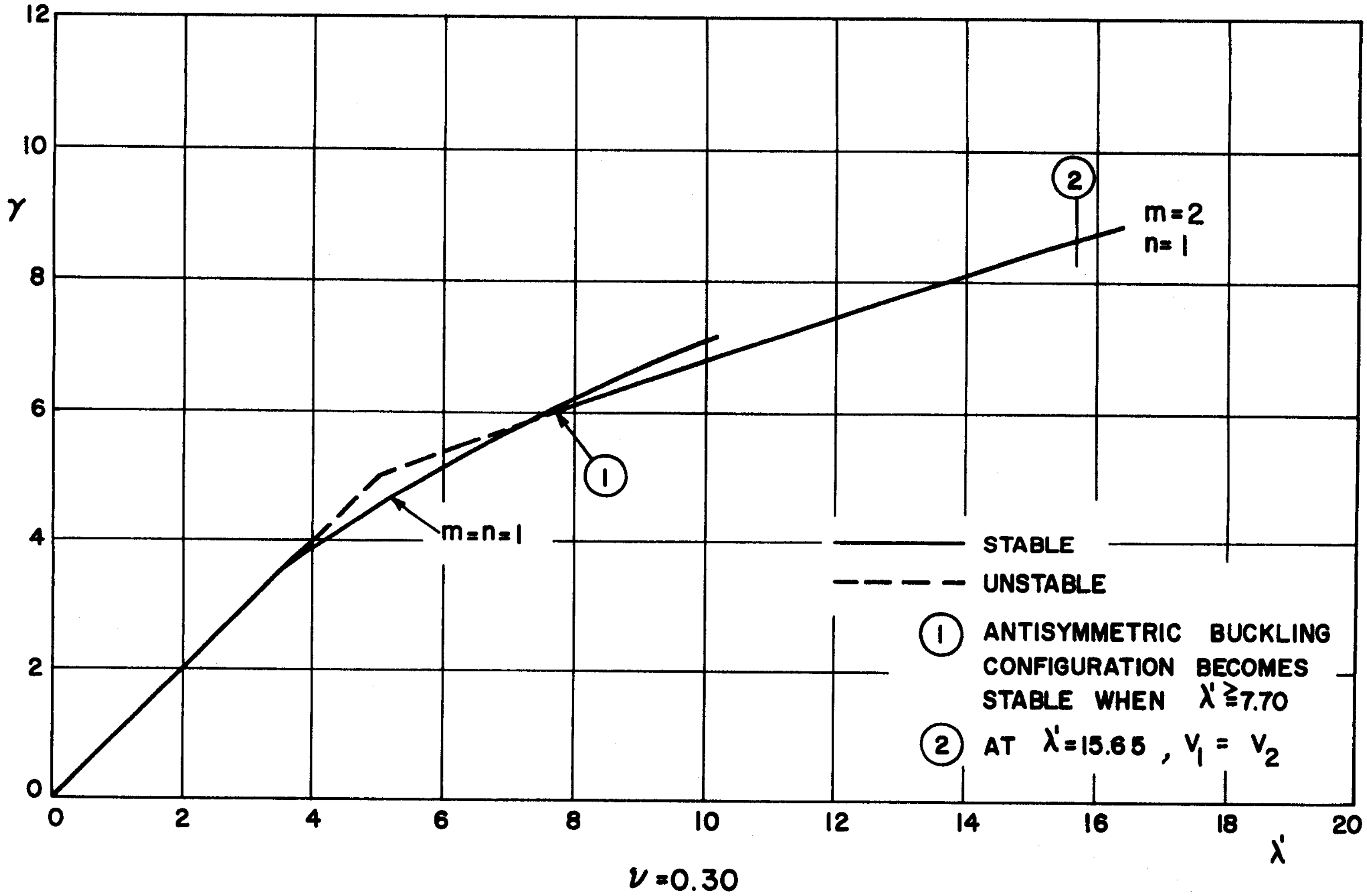


Figure 7. Nondimensional Load-Shortening Curves for Rectangular Plate under Hydrostatic Pressure a/b = 2.45.

conclusion that the antisymmetric buckling configuration for a square plate remains unstable. Figures 6 and 7 treat the rectangular plates of aspect ratio 2 and 2.45, respectively. It is interesting to note that a new equilibrium configuration (not shown) becomes possible at the value of λ' at which the antisymmetric buckling configuration becomes stable. This secondary bifurcation and the unstable character of the new configuration can be shown by considering the first and second variations of the potential energy. The value of λ' associated with equal potential energies for the two stable buckled states is also indicated in Figure 7.

The possibility of secondary buckling from the fundamental mode into a yet higher mode ($m = n = 2$) is treated in Figure 8, which shows the μ' versus λ' curves of a square plate subjected to plane hydrostatic pressure after the plate has been forced to buckle into that mode. Only the two vibration modes $p = q = 1$ and $p = 2$, $q = 1$ produce negative values of μ'. It is noted that these values remain negative; hence for a square plate the buckling configuration $m = n = 2$ is also unstable. This is confirmed by the load-shortening curves of the $m = n = 1$ and $m = n = 2$ buckling configurations shown in Figure 5. Since the two curves do not intersect the possibility of snap-through from the symmetric ($m = n = 1$) buckling configuration into the antisymmetric ($m = n = 2$) buckling configuration is ruled out.

The behavior of plates subjected to uniaxial edge compression is radically different. This has been treated by Stein[18] and others and is corroborated in Figures 9 and 10. In this case even a square plate exhibits a stable antisymmetric ($m = 2$, $n = 1$) equilibrium configuration when λ'' becomes sufficiently large. This change-over from

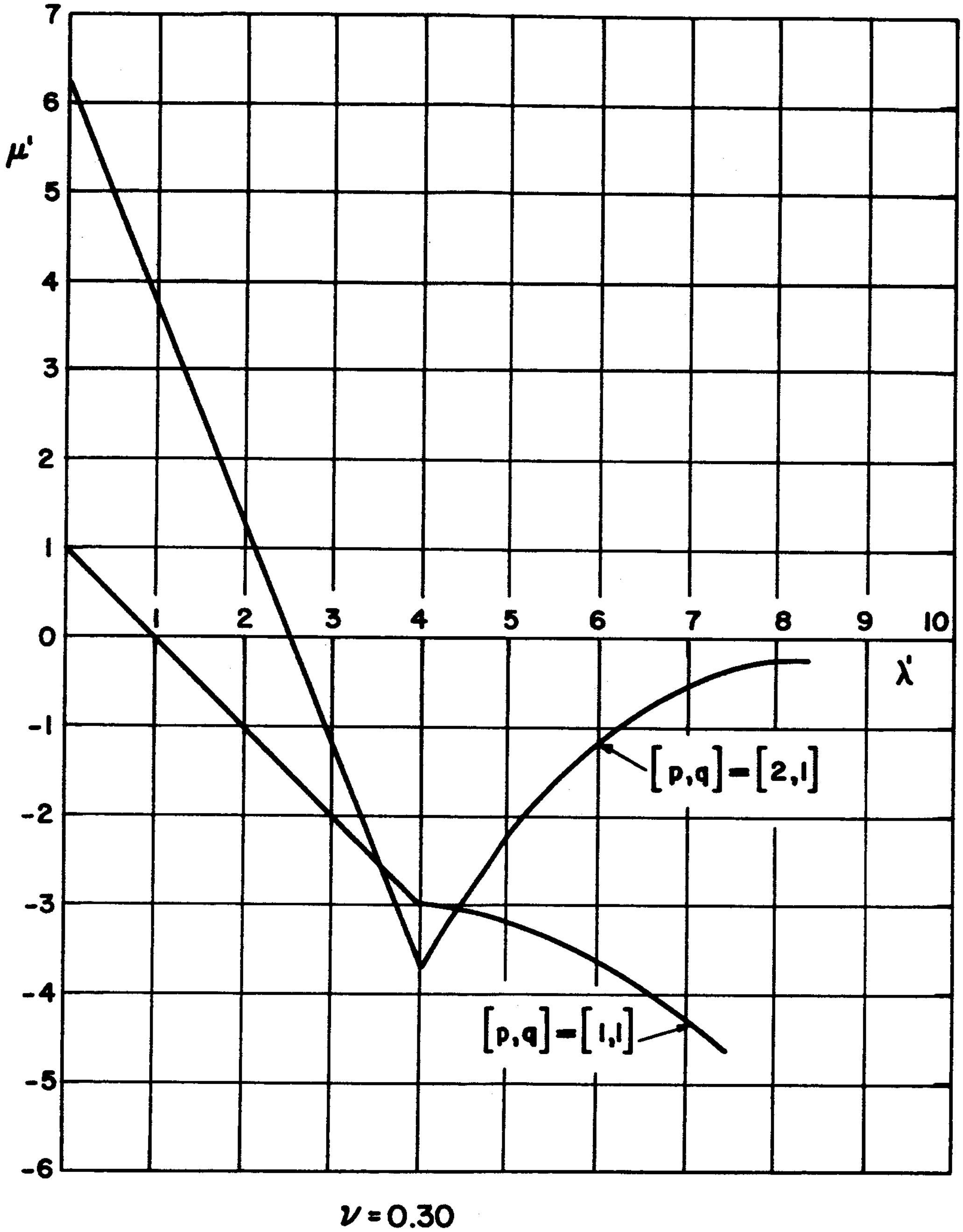
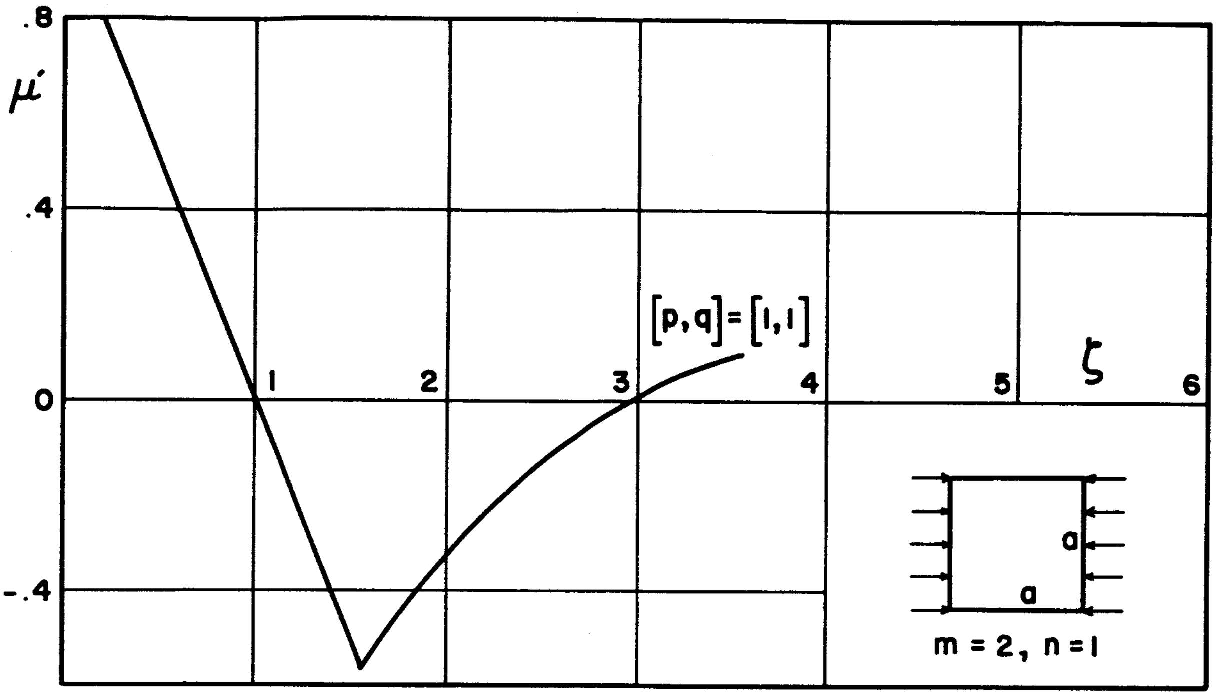
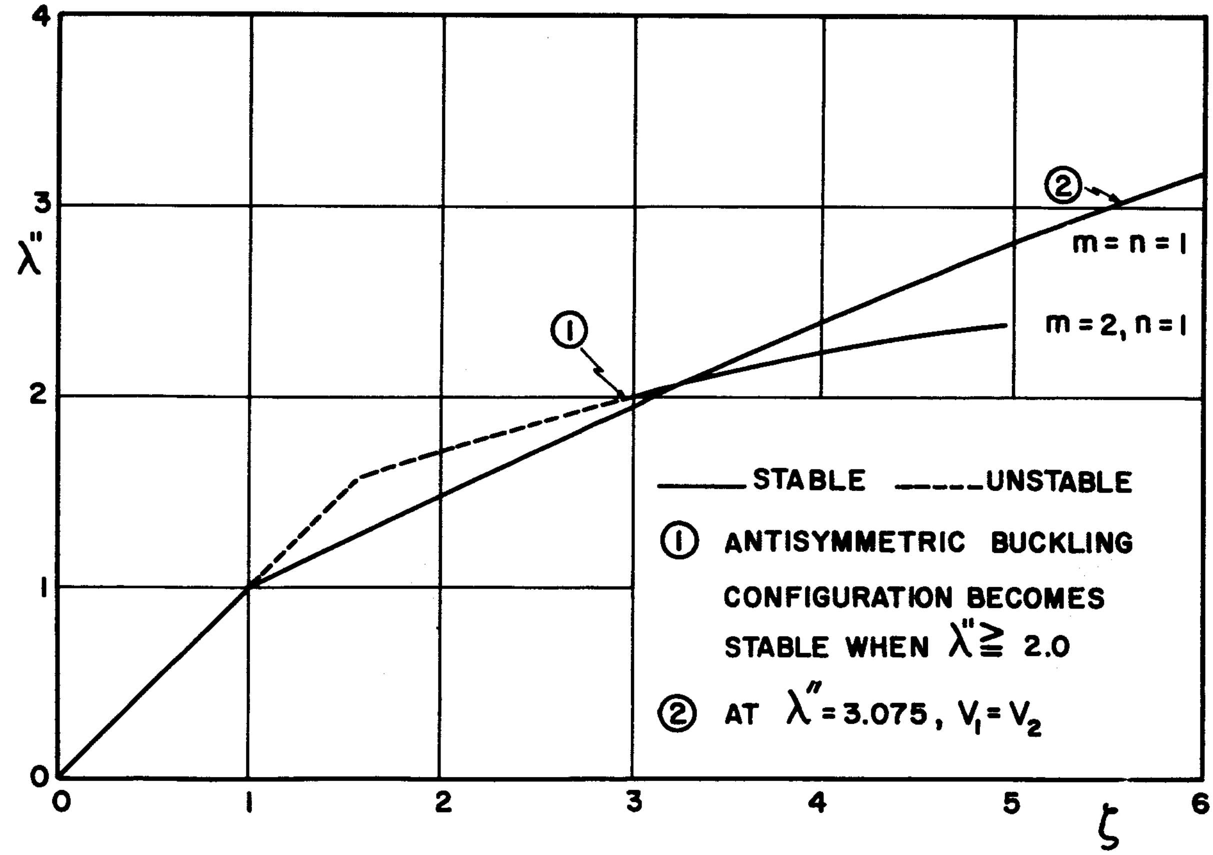


Figure 8. Nondimensional Frequency Squared-Load Curves for Square Plate under Hydrostatic Pressure m = n = 2 Buckling Mode.

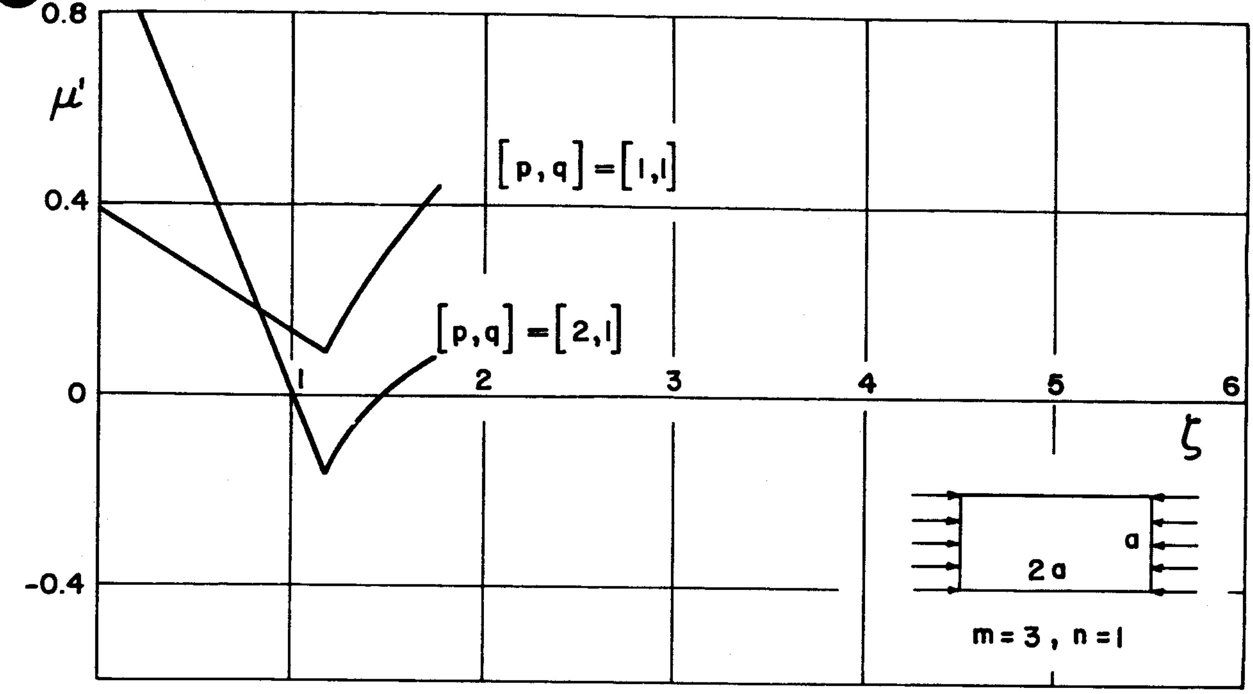


(a) Nondimensional Frequency Squared-Load Curve

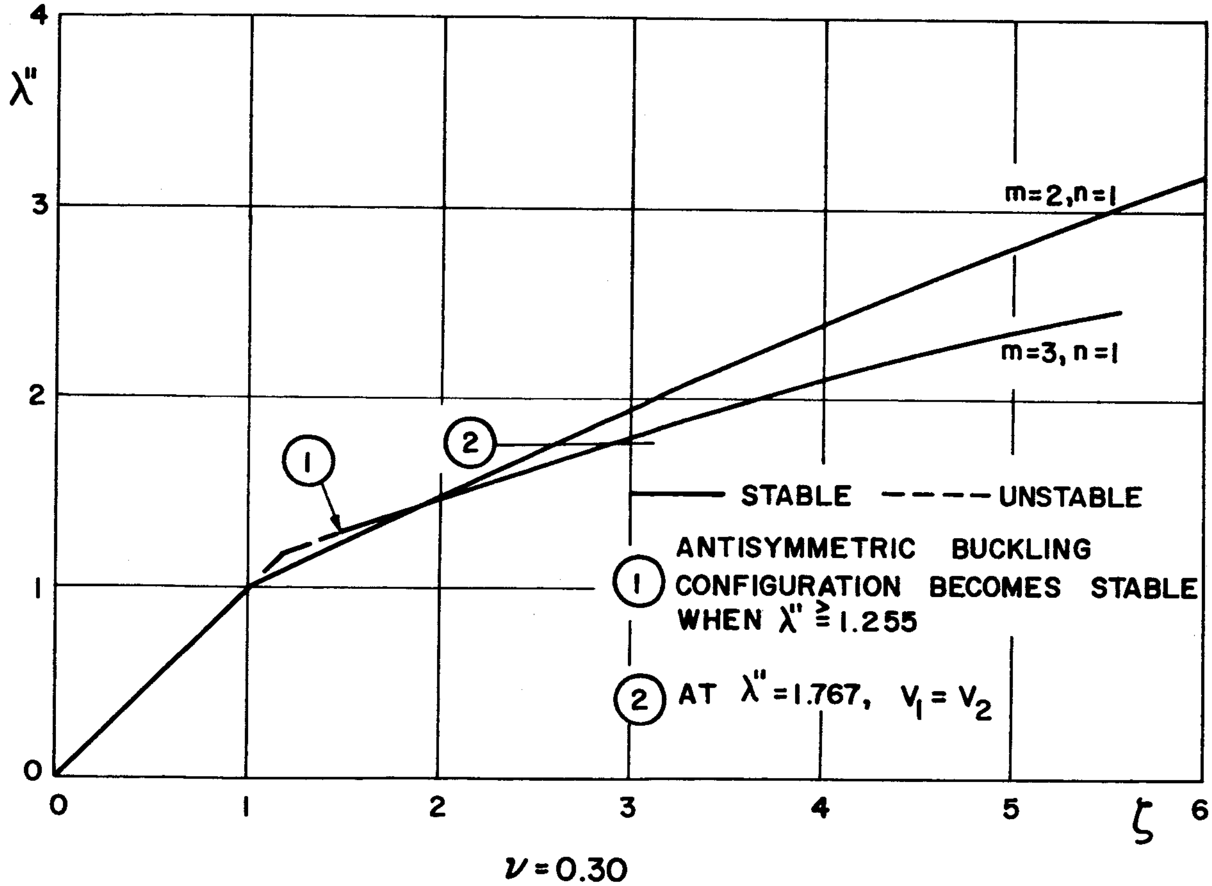


(b) Nondimensional Load-Shortening Curves

Figure 9. Square Plate - Uniaxial Edge Compression.



(a) Nondimensional Frequency Squared-Load Curve



(b) Nondimensional Load-Shortening Curves

Figure 10. Rectangular Plate - Uniaxial Edge Compression a/b = 2.

instability to stability is again accompanied by the emergence of a new unstable configuration (again not indicated in the figures). It is noted also that the values of λ'' so defined (representing lower bounds to secondary buckling) as well as the values of λ'' associated with equal potential energies are very much smaller than in the case of hydrostatic pressure. This is in good qualitative agreement with reported test results.

CHAPTER VI

CONCLUSIONS

It has been demonstrated that perturbation techniques can be used effectively to analyze the dynamic behavior of rectangular plates after they have buckled. The ensuing series show satisfactory convergence for a technically significant range of the load parameter.

Natural frequencies of vibration have been shown to be extremely sensitive to buckling amplitudes, displaying the most pronounced increase in connection with the symmetric vibratory mode. For sufficiently large load parameters this mode, which is primarily extensional, ceases to be associated with the longest period of vibration; however, it becomes more nearly inextensional as buckling proceeds and may therefore again return to its previous fundamental position.

The stability of higher buckling configurations has been investigated by studying the real or imaginary character of the frequencies of vibration about these configurations. The results indicate that all plates under uniaxial edge compression, and rectangular plates of sufficiently large aspect ratio under hydrostatic edge pressure, may eventually exhibit stable secondary buckling modes. The concomitant load parameters represent lower bounds to "secondary buckling loads" which signify the possibility of a sudden snap-through from one buckling configuration into another. This phenomenon had been widely observed before; the present calculations tend to conform with previously reported experimental results.

APPENDIX A

LIST OF FUNCTIONS - HYDROSTATIC EDGE PRESSURE

(1) Static Functions

(a) Deflection

$$W = \varepsilon W^{(1)} + \varepsilon^3 W^{(3)} + \varepsilon^5 W^{(5)} + \cdots$$

in which

$$W^{(1)} = h \sin\frac{m\pi}{a}x \, \sin\frac{n\pi}{b}y$$

$$W^{(3)} = h\left[A_4 \sin\frac{m\pi}{a}x \, \sin\frac{3n\pi}{b}y + \bar{A}_4 \sin\frac{3m\pi}{a}x \, \sin\frac{n\pi}{b}y\right]$$

$$\begin{aligned} W^{(5)} = h\Big[& A_{51} \sin\frac{m\pi}{a}x \, \sin\frac{3n\pi}{b}y + \bar{A}_{51} \sin\frac{3m\pi}{a}x \, \sin\frac{n\pi}{b}y + A_{52} \sin\frac{3m\pi}{a}x \, \sin\frac{3n\pi}{b}y \\ & + A_{53} \sin\frac{5m\pi}{a}x \, \sin\frac{n\pi}{b}y + \bar{A}_{53} \sin\frac{m\pi}{a}x \, \sin\frac{5n\pi}{b}y \\ & + A_{54} \sin\frac{5m\pi}{a}x \, \sin\frac{3n\pi}{b}y + \bar{A}_{54} \sin\frac{3m\pi}{a}x \, \sin\frac{5n\pi}{b}y \Big] \end{aligned}$$

with

$$A_4 = \frac{Eh^3}{128D} \frac{\left(\frac{m\pi}{a}\right)^4}{\left(\frac{n\pi}{b}\right)^2\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{3n\pi}{b}\right)^2\right]}$$

$$\bar{A}_4 = \frac{Eh^3}{128D} \frac{\left(\frac{n\pi}{b}\right)^4}{\left(\frac{m\pi}{a}\right)^2\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{3m\pi}{a}\right)^2\right]}$$

$$\begin{aligned} A_{51} = (A_4)^2 \Bigg\{ & \frac{\left(\frac{n\pi}{b}\right)^2}{(1+\nu)\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \left[(25+9\nu) - (24+8\nu)\frac{\left(\frac{n\pi}{b}\right)^2}{\left(\frac{m\pi}{a}\right)^2}\right] + \frac{\left(\frac{m\pi}{a}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \\ & - 4 - \frac{\left(\frac{2n\pi}{b}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} - \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]^2} \Bigg\} - A_4\bar{A}_4 \frac{\left(\frac{2n\pi}{b}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \end{aligned}$$

$$\begin{aligned} \bar{A}_{51} = (\bar{A}_4)^2 \Bigg\{ & \frac{\left(\frac{m\pi}{a}\right)^2}{(1+\nu)\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]} \left[(25+9\nu) - (24+8\nu)\frac{\left(\frac{m\pi}{a}\right)^2}{\left(\frac{n\pi}{b}\right)^2}\right] + \frac{\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]} \\ & - 4 - \frac{\left(\frac{2m\pi}{a}\right)^4}{\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]^2} - \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \Bigg\} - A_4\bar{A}_4 \frac{\left(\frac{2m\pi}{a}\right)^4}{\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]^2} \end{aligned}$$

$$A_{52} = A_4 \frac{Eh^3}{128D} \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \left\{ 1 + \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} \right\}$$

$$+ \bar{A}_4 \frac{Eh^3}{128D} \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \left\{ 1 + \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \right\}$$

$$A_{53} = \bar{A}_4 \frac{Eh^3}{128D} \frac{\left(\frac{n\pi}{b}\right)^4}{\left(\frac{m\pi}{a}\right)^2\left[\left(\frac{5m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]} \left\{ 1 + \frac{3\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \right\}$$

$$\bar{A}_{53} = A_4 \frac{Eh^3}{128D} \frac{\left(\frac{m\pi}{a}\right)^4}{\left(\frac{n\pi}{b}\right)^2\left[\left(\frac{5n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]} \left\{ 1 + \frac{3\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} \right\}$$

$$A_{54} = -A_4 \frac{Eh^3}{128D} \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{5m\pi}{a}\right)^2+\left(\frac{3n\pi}{b}\right)^2\right]\left[3\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]} \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$$

$$\bar{A}_{54} = -A_4 \frac{Eh^3}{128D} \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{5n\pi}{b}\right)^2+\left(\frac{3m\pi}{a}\right)^2\right]\left[3\left(\frac{n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]} \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2}$$

(b) Additional Membrane Stresses

$$T_{ij}' = \varepsilon^2 T_{ij}^{(2)} + \varepsilon^4 T_{ij}^{(4)} + \cdots\cdots$$

in which

$$T_{xx}^{(2)} = \frac{Eh^2}{8(1-\nu^2)} \left[\left(\frac{m\pi}{a}\right)^2 + \nu\left(\frac{n\pi}{b}\right)^2 - (1-\nu^2)\left(\frac{m\pi}{a}\right)^2 \cos\frac{2n\pi}{b}y \right]$$

$$T_{yy}^{(2)} = \frac{Eh^2}{8(1-\nu^2)} \left[\left(\frac{n\pi}{b}\right)^2 + \nu\left(\frac{m\pi}{a}\right)^2 - (1-\nu^2)\left(\frac{n\pi}{b}\right)^2 \cos\frac{2m\pi}{a}x \right]$$

$$T_{xy}^{(2)} = 0$$

$$T_{xx}^{(4)} = \frac{Eh^2}{4}\left(\frac{m\pi}{a}\right)^2\Big\{ A_4 \cos\frac{2n\pi}{b}y - A_4 \cos\frac{4n\pi}{b}y$$

$$- \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} 4(A_4+\bar{A}_4)\cos\frac{2m\pi}{a}x\cos\frac{2n\pi}{b}y$$

$$+ \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \bar{A}_4 \cos\frac{4m\pi}{a}x\cos\frac{2n\pi}{b}y$$

$$+ \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} 4A_4 \cos\frac{2m\pi}{a}x\cos\frac{4n\pi}{b}y \Big\}$$

$$T_{yy}^{(4)} = \frac{Eh^2}{4}\left(\frac{n\pi}{b}\right)^2\Big\{ \bar{A}_4 \cos\frac{2m\pi}{a}x - \bar{A}_4 \cos\frac{4m\pi}{a}x$$

$$- \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} 4(\bar{A}_4+A_4)\cos\frac{2m\pi}{a}x\cos\frac{2n\pi}{b}y$$

$$+ \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} A_4 \cos\frac{2m\pi}{a}x\cos\frac{4n\pi}{b}y$$

$$+ \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} 4\bar{A}_4 \cos\frac{4m\pi}{a}x\cos\frac{2n\pi}{b}y \Big\}$$

$$T_{xy}^{(4)} = \frac{Eh^2}{4}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\Big\{ - \frac{\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} 4(A_4+A_4)\sin\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}y$$

$$+ \frac{\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} 2\bar{A}_4 \sin\frac{4m\pi}{a}x\sin\frac{2n\pi}{b}y$$

$$+ \frac{\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} 2A_4 \sin\frac{2m\pi}{a}x\sin\frac{4n\pi}{b}y \Big\}$$

(c) Load Parameter

$$\lambda = \lambda_0 + \varepsilon^2\lambda_2 + \varepsilon^4\lambda_4 + \cdots$$

in which

$$\lambda_0 = \frac{D}{h}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$\lambda_2 = \frac{Eh^2}{16(1-\nu^2)}\,\frac{1}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\left\{(3-\nu^2)\left[\left(\frac{m\pi}{a}\right)^4+\left(\frac{n\pi}{b}\right)^4\right]+4\nu\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right\}$$

$$\lambda_4 = -\frac{3Eh^2}{16}\,\frac{1}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\left[\left(\frac{m\pi}{a}\right)^4 A_4 + \left(\frac{n\pi}{b}\right)^4 \bar{A}_4\right]$$

(d) Additional Membrane Displacements

$$U' = \varepsilon^2 U^{(2)} + \varepsilon^4 U^{(4)} + \cdots$$

$$V' = \varepsilon^2 V^{(2)} + \varepsilon^4 V^{(4)} + \cdots$$

in which

$$U^{(2)} = \frac{h^2}{16\left(\frac{m\pi}{a}\right)}\left[-\left(\frac{m\pi}{a}\right)^2 + \nu\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\cos\frac{2n\pi}{b}y\right]\sin\frac{2m\pi}{a}x$$

$$V^{(2)} = \frac{h^2}{16\left(\frac{n\pi}{b}\right)}\left[-\left(\frac{n\pi}{b}\right)^2 + \nu\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\cos\frac{2m\pi}{a}x\right]\sin\frac{2n\pi}{b}y$$

$$U^{(4)} = G_1\sin\frac{2m\pi}{a}x + G_2\sin\frac{4m\pi}{a}x + G_3\sin\frac{2m\pi}{a}x\cos\frac{2n\pi}{b}y$$
$$+ G_4\sin\frac{4m\pi}{a}x\cos\frac{2n\pi}{b}y + G_5\sin\frac{2m\pi}{a}x\cos\frac{4n\pi}{b}y$$

$$V^{(4)} = \bar{G}_1\sin\frac{2n\pi}{b}y + \bar{G}_2\sin\frac{4n\pi}{b}y + \bar{G}_3\cos\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}y$$
$$+ \bar{G}_4\cos\frac{2m\pi}{a}x\sin\frac{4n\pi}{b}y + \bar{G}_5\cos\frac{4m\pi}{a}x\sin\frac{2n\pi}{b}y$$

with

$$G_1 = -\frac{h^2}{16\left(\frac{m\pi}{a}\right)}\left[6\left(\frac{m\pi}{a}\right)^2 + 2\nu\left(\frac{n\pi}{b}\right)^2\right]\bar{A}_4$$

$$\bar{G}_1 = -\frac{h^2}{16\left(\frac{n\pi}{b}\right)}\left[6\left(\frac{n\pi}{b}\right)^2 + 2\nu\left(\frac{m\pi}{a}\right)^2\right]A_4$$

$$G_2 = -\frac{h^2}{16(\frac{m\pi}{a})}\left[3(\frac{m\pi}{a})^2 - \nu(\frac{n\pi}{b})^2\right]\bar{A}_4$$

$$\bar{G}_2 = -\frac{h^2}{16(\frac{n\pi}{b})}\left[3(\frac{n\pi}{b})^2 - \nu(\frac{m\pi}{a})^2\right]A_4$$

$$G_3 = \frac{h^2(\frac{m\pi}{a})}{8[(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2]^2}\Big\{\left[-5(\frac{n\pi}{b})^4 - 2(1-2\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 - (\frac{m\pi}{a})^4\right]A_4$$

$$+\left[-(\frac{n\pi}{b})^4 + 2(3+2\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + 3(\frac{m\pi}{a})^4\right]\bar{A}_4\Big\}$$

$$\bar{G}_3 = \frac{h^2(\frac{n\pi}{b})}{8[(\frac{n\pi}{b})^2+(\frac{m\pi}{a})^2]^2}\Big\{\left[-5(\frac{m\pi}{a})^4 - 2(1-2\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 - (\frac{n\pi}{b})^4\right]\bar{A}_4$$

$$+\left[-(\frac{m\pi}{a})^4 + 2(3+2\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + 3(\frac{n\pi}{b})^4\right]A_4\Big\}$$

$$G_4 = \frac{h^2(\frac{m\pi}{a})}{8[(\frac{2m\pi}{a})^2+(\frac{n\pi}{b})^2]^2}\left[2(\frac{n\pi}{b})^4 + 2(6-\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + 24(\frac{m\pi}{a})^4\right]\bar{A}_4$$

$$\bar{G}_4 = \frac{h^2(\frac{n\pi}{b})}{8[(\frac{2n\pi}{b})^2+(\frac{m\pi}{a})^2]^2}\left[2(\frac{m\pi}{a})^4 + 2(6-\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + 24(\frac{n\pi}{b})^4\right]A_4$$

$$G_5 = \frac{h^2(\frac{m\pi}{a})}{8[(\frac{2n\pi}{b})^2+(\frac{m\pi}{a})^2]^2}\left[20(\frac{n\pi}{b})^4 + (8-\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + (\frac{m\pi}{a})^4\right]A_4$$

$$\bar{G}_5 = \frac{h^2(\frac{n\pi}{b})}{8[(\frac{2m\pi}{a})^2+(\frac{n\pi}{b})^2]^2}\left[20(\frac{m\pi}{a})^4 + (8-\nu)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + (\frac{n\pi}{b})^4\right]\bar{A}_4$$

(2) Dynamic Functions

(A) Static Buckling Configuration m = 1 and n = 1

(i) Vibration Mode p = 1 and q = 1

(a) Deflection

$$w = w^{(0)} + \varepsilon^2 w^{(2)} + \varepsilon^4 w^{(4)} + \cdots$$

in which

$$w^{(0)} = \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y$$

$$w^{(2)} = 3\left[A_4 \sin\frac{\pi}{a}x \sin\frac{3\pi}{b}y + \bar{A}_4 \sin\frac{3\pi}{a}x \sin\frac{\pi}{b}y\right]$$

$$w^{(4)} = 5\left[A_{51} \sin\frac{\pi}{a}x \sin\frac{3\pi}{b}y + \bar{A}_{51} \sin\frac{3\pi}{a}x \sin\frac{\pi}{b}y\right.$$

$$+ A_{52} \sin\frac{3\pi}{a}x \sin\frac{3\pi}{b}y + A_{53} \sin\frac{5\pi}{a}x \sin\frac{\pi}{b}y$$

$$+ \bar{A}_{53} \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y + A_{54} \sin\frac{5\pi}{a}x \sin\frac{3\pi}{b}y$$

$$\left. + \bar{A}_{54} \sin\frac{3\pi}{a}x \sin\frac{5\pi}{b}y\right]$$

$$+ 2\lambda_2 h \left\{ \frac{\left[\left(\frac{\pi}{a}\right)^2 - 3\left(\frac{\pi}{b}\right)^2\right] A_4}{\left(\frac{2\pi}{b}\right)^3 \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{3\pi}{b}\right)^2\right] D} \sin\frac{\pi}{a}x \sin\frac{3\pi}{b}y \right.$$

$$\left. + \frac{\left[\left(\frac{\pi}{b}\right)^2 - 3\left(\frac{\pi}{a}\right)^2\right] \bar{A}_4}{\left(\frac{2\pi}{a}\right)^2 \left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{3\pi}{a}\right)^2\right] D} \sin\frac{3\pi}{a}x \sin\frac{\pi}{b}y \right\}$$

(b) Membrane Stresses

$$t_{ij}' = \varepsilon t_{ij}^{(1)} + \varepsilon^3 t_{ij}^{(3)} + \cdots$$

in which

$$t_{xx}^{(1)} = \frac{Eh}{4(1-\nu^2)}\left[\left(\frac{\pi}{a}\right)^2 + \nu\left(\frac{\pi}{b}\right)^2 - (1-\nu^2)\left(\frac{\pi}{a}\right)^2 \cos\frac{2\pi}{b}y\right]$$

$$t_{yy}^{(1)} = \frac{Eh}{4(1-\nu^2)}\left[\left(\frac{\pi}{b}\right)^2 + \nu\left(\frac{\pi}{a}\right)^2 - (1-\nu^2)\left(\frac{\pi}{b}\right)^2 \cos\frac{2\pi}{a}x\right]$$

$$t_{xy}^{(1)} = 0$$

$$t_{xx}^{(3)} = Eh\left(\frac{\pi}{a}\right)^2\left\{A_4\cos\frac{2\pi}{b}y - A_4\cos\frac{4\pi}{b}y - \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}4(A_4+\bar{A}_4)\cos\frac{2\pi}{a}x\cos\frac{2\pi}{b}y\right.$$

$$\left. + \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}\bar{A}_4\cos\frac{4\pi}{a}x\cos\frac{2\pi}{b}y + \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{2\pi}{b}\right)^2+\left(\frac{\pi}{a}\right)^2\right]^2}4A_4\cos\frac{2\pi}{a}x\cos\frac{4\pi}{b}y\right\}$$

$$t_{yy}^{(3)} = Eh\left(\frac{\pi}{b}\right)^2\left\{\bar{A}_4\cos\frac{2\pi}{a}x - \bar{A}_4\cos\frac{4\pi}{a}x - \frac{\left(\frac{\pi}{a}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}4(A_4+\bar{A}_4)\cos\frac{2\pi}{a}x\cos\frac{2\pi}{b}y\right.$$

$$\left. + \frac{\left(\frac{\pi}{a}\right)^4}{\left[\left(\frac{2\pi}{b}\right)^2+\left(\frac{\pi}{a}\right)^2\right]^2}A_4\cos\frac{2\pi}{a}x\cos\frac{4\pi}{b}y + \frac{\left(\frac{\pi}{a}\right)^4}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}4\bar{A}_4\cos\frac{4\pi}{a}x\cos\frac{2\pi}{b}y\right\}$$

$$t_{xy}^{(3)} = Eh\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)\left\{-\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}4(A_4+\bar{A}_4)\sin\frac{2\pi}{a}x\sin\frac{2\pi}{b}y\right.$$

$$\left. + \frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}2\bar{A}_4\sin\frac{4\pi}{a}x\sin\frac{2\pi}{b}y + \frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{2\pi}{b}\right)^2+\left(\frac{\pi}{a}\right)^2\right]^2}2A_4\sin\frac{2\pi}{a}x\sin\frac{4\pi}{b}y\right\}$$

(c) Frequency Parameter

$$\mu = \mu^{(0)} + \varepsilon^2\mu^{(2)} + \varepsilon^4\mu^{(4)} + \cdots$$

in which

$$\mu^{(0)} = 0$$

$$\mu^{(2)} = \frac{Eh^2}{8(1-\nu^2)}\left\{(3-\nu^2)\left[\left(\frac{\pi}{a}\right)^4+\left(\frac{\pi}{b}\right)^4\right] + 4\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2\right\}$$

$$\mu^{(4)} = -\frac{3Eh^2}{4}\left[\left(\frac{\pi}{a}\right)^4A_4 + \left(\frac{\pi}{b}\right)^4\bar{A}_4\right]$$

(ii) Vibration Mode p = 2 and q = 1

(a) Deflection

$$w = w^{(0)} + \varepsilon^2 w^{(2)} + \varepsilon^4 w^{(4)} + \cdots$$

in which

$$w^{(0)} = \sin\frac{2\pi}{a}x\,\sin\frac{\pi}{b}y$$

$$w^{(2)} = A_{66}\sin\frac{2\pi}{a}x\,\sin\frac{3\pi}{b}y + A_{68}\sin\frac{4\pi}{a}x\,\sin\frac{\pi}{b}y + A_{69}\sin\frac{4\pi}{a}x\,\sin\frac{3\pi}{b}y$$

$$w^{(4)} = A_{324}\sin\frac{2\pi}{a}x\,\sin\frac{3\pi}{b}y + A_{325}\sin\frac{2\pi}{a}x\,\sin\frac{5\pi}{b}y + A_{326}\sin\frac{4\pi}{a}x\,\sin\frac{\pi}{b}y$$

$$+ A_{327}\sin\frac{4\pi}{a}x\,\sin\frac{3\pi}{b}y + A_{328}\sin\frac{4\pi}{a}x\,\sin\frac{5\pi}{b}y + A_{329}\sin\frac{6\pi}{a}x\,\sin\frac{\pi}{b}y$$

$$+ A_{330}\sin\frac{6\pi}{a}x\,\sin\frac{3\pi}{b}y + A_{331}\sin\frac{6\pi}{a}x\,\sin\frac{5\pi}{b}y$$

with

$$A_{66} = \frac{Eh^3}{128D}\,\frac{\left\{\dfrac{9\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2} + \dfrac{25\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2} + 4\left(\frac{\pi}{a}\right)^4\right\}}{\left(\frac{\pi}{b}\right)^2\left[7\left(\frac{\pi}{a}\right)^2+9\left(\frac{\pi}{b}\right)^2\right]}$$

$$A_{68} = \frac{Eh^3}{192D}\,\frac{\left\{3\left(\frac{\pi}{b}\right)^4 + \dfrac{25\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\right\}}{\left(\frac{\pi}{a}\right)^2\left[19\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]}$$

$$A_{69} = -\frac{Eh^3}{64D}\,\frac{\dfrac{\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}}{\left[3\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]\left[19\left(\frac{\pi}{a}\right)^2+9\left(\frac{\pi}{b}\right)^2\right]}$$

$$A_{324} = \frac{Eh^3}{128D}\,\frac{\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left(\frac{\pi}{b}\right)^2\left[7\left(\frac{\pi}{a}\right)^2+9\left(\frac{\pi}{b}\right)^2\right]}\Bigg\{A_4\Bigg[-\frac{16}{\left(\frac{\pi}{b}\right)^4} - \frac{36}{\left(\frac{\pi}{a}\right)^4} - \frac{1225}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}$$

$$- \frac{1225}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2} - \frac{25}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2} - \frac{25}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\Bigg] + \bar{A}_4\Bigg[\frac{466}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}$$

$$+ \frac{16}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}\Bigg] + A_{66}\Bigg[\frac{\left[\left(\frac{\pi}{a}\right)^4+\left(\frac{\pi}{b}\right)^4\right]}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]}\,\frac{8}{\left(\frac{\pi}{b}\right)^2\left(\frac{\pi}{a}\right)^4} + \frac{\left[\left(\frac{\pi}{a}\right)^2-\left(\frac{\pi}{b}\right)^2\right]}{(1+\nu)\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]}\,\frac{16}{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}$$

$$+\frac{4}{(\frac{\pi}{b})^4}+\frac{4}{(\frac{\pi}{a})^4}+\frac{80}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}-\frac{625}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}-\frac{624}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$-\frac{1}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big]-A_{68}\frac{625}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+A_{69}\left(\frac{9}{(\frac{\pi}{a})^4}+\frac{25}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{49}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\right)\Bigg\}$$

$$A_{325}=\frac{Eh^3}{384D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{(\frac{\pi}{b})^2[7(\frac{\pi}{a})^2+25(\frac{\pi}{b})^2]}\Bigg\{A_4\Big[\frac{8}{(\frac{\pi}{b})^4}+\frac{441}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}+\frac{1225}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}$$

$$+\frac{9}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{121}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]+A_{66}\Big[\frac{4}{(\frac{\pi}{b})^4}+\frac{225}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}+\frac{49}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big]$$

$$-A_{69}\frac{2401}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Bigg\}$$

$$A_{326}=\frac{Eh^3}{192D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{(\frac{\pi}{a})^2[19(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}\Bigg\{A_4\left(-\frac{36}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}-\frac{1346}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\right)$$

$$+\bar{A}_4\left(-\frac{36}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}-\frac{9}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}-\frac{441}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}-\frac{6}{(\frac{\pi}{a})^4}\right)$$

$$-A_{66}\frac{625}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+A_{68}\Bigg(\frac{[(\frac{\pi}{a})^4+(\frac{\pi}{b})^4]}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}\frac{12}{(\frac{\pi}{a})^2(\frac{\pi}{b})^4}-\frac{[(\frac{\pi}{a})^2-(\frac{\pi}{b})^2]}{(1+\nu)[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}\frac{24}{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}$$

$$-\frac{12}{(\frac{\pi}{b})^4}-\frac{624}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}-\frac{81}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{81}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Bigg)$$

$$+A_{69}\left(\frac{16}{(\frac{\pi}{b})^4}+\frac{25}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{441}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\right)\Bigg\}$$

$$A_{327}=\frac{Eh^3}{64D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[57(\frac{\pi}{a})^4+65(\frac{\pi}{a})^2(\frac{\pi}{b})^2+18(\frac{\pi}{b})^4]}\Bigg\{A_4\Big(\frac{18}{(\frac{\pi}{a})^4}+\frac{4}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}+\frac{49}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+\frac{1225}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}+\frac{25}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]+\bar{A}_4\Big(\frac{4}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}+\frac{49}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+\frac{225}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]+A_{66}\Big[\frac{9}{(\frac{\pi}{a})^4}+\frac{25}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{49}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big]$$

$$+A_{68}\Big(\frac{16}{(\frac{\pi}{b})^4}+\frac{25}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}+\frac{441}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big)$$

$$+A_{69}\Big(\frac{[(\frac{\pi}{a})^4+(\frac{\pi}{b})^4][12(\frac{\pi}{a})^2+8(\frac{\pi}{b})^2]}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2](\frac{\pi}{a})^4(\frac{\pi}{b})^4}-\frac{8[(\frac{\pi}{a})^2-(\frac{\pi}{b})^2]}{(1+\nu)[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2](\frac{\pi}{a})^2(\frac{\pi}{b})^2}$$

$$+\frac{4}{(\frac{\pi}{b})^4}+\frac{4}{(\frac{\pi}{a})^4}-\frac{2401}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}-\frac{2401}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}-\frac{1}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}$$

$$+\frac{81}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]\Big\}.$$

$$A_{328}=\frac{Eh^3}{192D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[19(\frac{\pi}{a})^4+63(\frac{\pi}{a})^2(\frac{\pi}{b})^2+50(\frac{\pi}{b})^4]}\Big\{A_4\Big[-\frac{9}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$-\frac{25}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}-\frac{49}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]-A_{66}\frac{1}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}$$

$$+A_{69}\Big[\frac{16}{(\frac{\pi}{b})^4}+\frac{49}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}+\frac{81}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big]\Big\}.$$

$$A_{329}=\frac{Eh^3}{512D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{(\frac{\pi}{a})^2[39(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}\Big\{\bar{A}_4\Big(\frac{6}{(\frac{\pi}{a})^4}+\frac{16}{[(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]^2}+\frac{49}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+\frac{81}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]+A_{68}\Big[\frac{3}{(\frac{\pi}{a})^4}+\frac{441}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]-A_{69}\frac{2401}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big\}.$$

$$A_{330}=\frac{Eh^3}{192D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[104(\frac{\pi}{a})^4+60(\frac{\pi}{a})^2(\frac{\pi}{b})^2+9(\frac{\pi}{b})^4]}\Big\{\bar{A}_4\Big(-\frac{9}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$- \frac{9}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big] - A_{68}\frac{81}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+ A_{69}\Big[\frac{9}{(\frac{\pi}{a})^4} + \frac{441}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{81}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big]\Big\}$$

$$A_{331} = \frac{Eh^3}{128D}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[156(\frac{\pi}{a})^4+217(\frac{\pi}{a})^2(\frac{\pi}{b})^2+75(\frac{\pi}{b})^4]}\Big\{-A_{69}\frac{1}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\Big\}$$

(b) Membrane Stresses

$$t_{ij}' = \varepsilon t_{ij}^{(1)} + \varepsilon^3 t_{ij}^{(3)} + \cdots$$

in which

$$t_{xx}^{(1)} = \frac{Eh}{4}\Big\{\frac{36(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y + \frac{4(\frac{\pi}{a})^2(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y\Big\}$$

$$t_{yy}^{(1)} = \frac{Eh}{4}\Big\{(\frac{\pi}{b})^2\cos\frac{\pi}{a}x - (\frac{\pi}{b})^2\cos\frac{3\pi}{a}x - \frac{9(\frac{\pi}{a})^4(\frac{\pi}{b})^2}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+ \frac{9(\frac{\pi}{a})^4(\frac{\pi}{b})^2}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y\Big\}$$

$$t_{xy}^{(1)} = \frac{Eh}{4}\Big\{-\frac{18(\frac{\pi}{a})^3(\frac{\pi}{b})^3}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\sin\frac{\pi}{a}x\sin\frac{2\pi}{b}y + \frac{6(\frac{\pi}{a})^3(\frac{\pi}{b})^3}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\sin\frac{3\pi}{a}x\sin\frac{2\pi}{b}y\Big\}$$

$$t_{xx}^{(3)} = \frac{Eh}{4}(\frac{\pi}{a})^2\Big\{\frac{(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(100A_4-100\bar{A}_4+4A_{66})\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-784A_4-400A_{66})\cos\frac{\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-196A_4-100A_{66}-100A_{68}+4A_{69})\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(400A_4+16A_{66}-784A_{69}) \cos\frac{3\pi}{a}x \cos\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(4\bar{A}_4+36A_{68}-196A_{69}) \cos\frac{5\pi}{a}x \cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}\; 16A_{69} \cos\frac{5\pi}{a}x \cos\frac{4\pi}{b}y \Big\}$$

$$t_{yy}^{(3)} = \frac{Eh}{4}\left(\frac{\pi}{b}\right)^2 \Big\{ A_4 \cos\frac{\pi}{a}x + A_{68}\cos\frac{3\pi}{a}x - (\bar{A}_4+A_{68})\cos\frac{5\pi}{a}x$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(25A_4-25\bar{A}_4+A_{66}) \cos\frac{\pi}{a}x \cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-49A_4-25A_{66}) \cos\frac{\pi}{a}x \cos\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-441A_4-225A_{66}-225A_{68}+9A_{69}) \cos\frac{3\pi}{a}x \cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(225A_4+9A_{66}-441A_{69}) \cos\frac{3\pi}{a}x \cos\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(25\bar{A}_4+225A_{68}-1225A_{69}) \cos\frac{5\pi}{a}x \cos\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^4}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(25A_{69}) \cos\frac{5\pi}{a}x \cos\frac{4\pi}{b}y \Big\}$$

$$t_{xy}^{(3)} = \frac{Eh}{4}\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right) \Big\{ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(50A_4-50\bar{A}_4+2A_{66}) \sin\frac{\pi}{a}x \sin\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-196A_4-100A_{66})\sin\frac{\pi}{a}x\sin\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-294A_4-150A_{66}-150A_{68}+6A_{69})\sin\frac{3\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(300A_4+12A_{66}-588A_{69})\sin\frac{3\pi}{a}x\sin\frac{4\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(10\bar{A}_4+90A_{68}-490A_{69})\sin\frac{5\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+ \frac{(\frac{\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]}(20A_{69})\sin\frac{5\pi}{a}x\sin\frac{4\pi}{b}y \Big\}$$

(c) Frequency Parameter

$$\mu = \mu^{(0)} + \varepsilon^2\mu^{(2)} + \varepsilon^4\mu^{(4)} + \cdots$$

in which

$$\mu^{(0)} = \frac{D}{h}\,3(\frac{\pi}{a})^2[(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]$$

$$\mu^{(2)} = \frac{Eh^2}{16}\Big\{-\frac{[(\frac{\pi}{a})^4+(\frac{\pi}{b})^4][(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]} + \frac{6(\frac{\pi}{a})^2(\frac{\pi}{b})^2[(\frac{\pi}{a})^2-(\frac{\pi}{b})^2]}{(1+\nu)[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}$$

$$+4[(\frac{\pi}{a})^4+(\frac{\pi}{b})^4] + \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{81(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big\}$$

$$\mu^{(4)} = \frac{Eh}{16}\Big\{A_4\Big[-5(\frac{\pi}{a})^4 + \frac{9(\frac{\pi}{a})^6}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]} - \frac{450(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} - \frac{98(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]$$

$$+\bar{A}_4\Big[9(\frac{\pi}{b})^4 + \frac{9(\frac{\pi}{a})^2(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]} + \frac{450(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big] + A_{66}\Big[-4(\frac{\pi}{a})^4$$

$$- \frac{9(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} - \frac{25(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big] - A_{68}\Big[3(\frac{\pi}{b})^4 + \frac{25(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big]$$

$$+ A_{69}\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big\}$$

(B) Static Buckling Configuration m = 2 and n = 1

(i) Vibration Mode p = 1 and q = 1

(a) Deflection

$$w = w^{(0)} + \varepsilon^2 w^{(2)} + \cdots$$

in which

$$w^{(0)} = \sin\frac{\pi}{a}x\,\sin\frac{\pi}{b}y$$

$$w^{(2)} = A_{162}\sin\frac{3\pi}{a}x\,\sin\frac{\pi}{b}y - A_{164}\sin\frac{3\pi}{a}x\,\sin\frac{3\pi}{b}y + A_{166}\sin\frac{\pi}{a}x\,\sin\frac{3\pi}{b}y$$

$$+ A_{168}\sin\frac{5\pi}{a}x\,\sin\frac{\pi}{b}y + A_{169}\sin\frac{5\pi}{a}x\,\sin\frac{3\pi}{b}y$$

with

$$A_{162} = \frac{Eh^3}{128D}\,\frac{-3(\frac{\pi}{b})^4 - \frac{225(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}}{(\frac{\pi}{a})^2[6(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}$$

$$A_{164} = \frac{Eh^3}{384D}\,\frac{-\frac{81(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2][2(\frac{\pi}{a})^2+3(\frac{\pi}{b})^2]}$$

$$A_{166} = \frac{Eh^3}{128D}\,\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4\Big\{\frac{225}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{49}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{4}{(\frac{\pi}{b})^4}\Big\}}{(\frac{\pi}{b})^2[-2(\frac{\pi}{a})^2+9(\frac{\pi}{b})^2]}$$

$$A_{168} = \frac{Eh^3}{384D}\,\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4\Big\{\frac{3}{(\frac{\pi}{a})^4} + \frac{49}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\Big\}}{(\frac{\pi}{a})^2[22(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]}$$

$$A_{169} = \frac{Eh^3}{128D} \frac{\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4\left\{-\frac{1}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\right\}}{\left[3\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]\left[22\left(\frac{\pi}{a}\right)^2+9\left(\frac{\pi}{b}\right)^2\right]}$$

(b) Membrane Stresses

$$t'_{ij} = \varepsilon t_{ij}^{(1)} + \varepsilon^3 t_{ij}^{(3)} + \cdots\cdot$$

in which

$$t_{xx}^{(1)} = \frac{Eh}{4}\left[-\frac{36\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y + \frac{4\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y\right]$$

$$t_{yy}^{(1)} = \frac{Eh}{4}\left[-\frac{9\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y - \left(\frac{\pi}{b}\right)^2\cos\frac{3\pi}{a}x\right.$$

$$\left. + \frac{9\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y + \left(\frac{\pi}{b}\right)^2\cos\frac{\pi}{a}x\right]$$

$$t_{xy}^{(1)} = \frac{Eh}{4}\left[-\frac{18\left(\frac{\pi}{a}\right)^3\left(\frac{\pi}{b}\right)^3}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\sin\frac{\pi}{a}x\sin\frac{2\pi}{b}y + \frac{6\left(\frac{\pi}{a}\right)^3\left(\frac{\pi}{b}\right)^3}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\sin\frac{3\pi}{a}x\sin\frac{2\pi}{b}y\right]$$

$$t_{xx}^{(3)} = \frac{Eh}{4}\left(\frac{\pi}{a}\right)^2\left\{\frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(4A_{14}-100A_{62}-36A_{164}+100A_{166}\right)\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y\right.$$

$$+ \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(-400A_{14}+1296A_{164}-\cancel{784}A_{166}\right)\cos\frac{\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+ \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(-100A_{14}-196A_{166}-196A_{168}+4A_{169}\right)\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+ \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(16A_{14}+400A_{166}-1936A_{169}\right)\cos\frac{3\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+\frac{(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-196\bar{A}_{14}+4A_{162}+324A_{164})\cos\frac{5\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-144A_{164})\cos\frac{5\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+\frac{(\frac{\pi}{b})^4}{[(\frac{7\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(100\bar{A}_{14}+36A_{168}-484A_{169})\cos\frac{7\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{\pi}{b})^4}{[(\frac{7\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(16A_{169})\cos\frac{7\pi}{a}x\cos\frac{2\pi}{b}y\Big\}$$

$$t_{yy}^{(3)}=\frac{Eh}{4}(\frac{\pi}{b})^2\Big\{A_{162}\cos\frac{\pi}{a}x+A_{168}\cos\frac{3\pi}{a}x+(\bar{A}_{14}-A_{162})\cos\frac{5\pi}{a}x$$

$$-(\bar{A}_{14}+A_{168})\cos\frac{7\pi}{a}x+\frac{(\frac{\pi}{a})^2}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(A_{14}-25A_{162}-9A_{164}+25A_{166})\cos\frac{\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{\pi}{a})^4}{[(\frac{\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-25A_{14}+81A_{164}-49A_{166})\cos\frac{\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+\frac{(\frac{\pi}{a})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-225A_{14}-441A_{166}-441A_{168}+9A_{169})\cos\frac{3\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{\pi}{a})^4}{[(\frac{3\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(9A_{14}+225A_{166}-1089A_{169})\cos\frac{3\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+\frac{(\frac{\pi}{a})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-1225\bar{A}_{14}+25A_{162}+2025A_{164})\cos\frac{5\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{\pi}{a})^4}{[(\frac{5\pi}{a})^2+(\frac{4\pi}{b})^2]^2}(-225A_{164})\cos\frac{5\pi}{a}x\cos\frac{4\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^4}{\left[\left(\frac{7\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(1225\bar{A}_{14}+441A_{168}-5929A_{169}\right)\cos\frac{7\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^4}{\left[\left(\frac{7\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(49A_{169}\right)\cos\frac{7\pi}{a}x\cos\frac{4\pi}{b}y\Bigg\}$$

$$t_{xy}^{(3)}=\frac{Eh}{4}\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)\Bigg\{\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(2A_{14}-50A_{162}-18A_{164}+50A_{166}\right)\sin\frac{\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(-100A_{14}+324A_{164}-196A_{166}\right)\sin\frac{\pi}{a}x\sin\frac{4\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]}\left(-150A_{14}-294A_{166}-294A_{168}+6A_{169}\right)\sin\frac{3\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{3\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(12A_{14}+300A_{166}-1452A_{169}\right)\sin\frac{3\pi}{a}x\sin\frac{4\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{5\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(-490\bar{A}_{14}+10A_{162}+810A_{164}\right)\sin\frac{5\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{5\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]^2}\left(-180A_{164}\right)\sin\frac{5\pi}{a}x\sin\frac{4\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{7\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]^2}\left(350\bar{A}_{14}+126A_{168}-1694A_{169}\right)\sin\frac{7\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+\frac{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}{\left[\left(\frac{7\pi}{a}\right)^2+\left(\frac{4\pi}{b}\right)^2\right]}\left(28A_{169}\right)\sin\frac{7\pi}{a}x\sin\frac{4\pi}{b}y$$

with

$$A_{14}=\frac{Eh^3}{128D}\frac{16\left(\frac{\pi}{a}\right)^4}{\left(\frac{\pi}{b}\right)^2\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{3\pi}{b}\right)^2\right]}\qquad\bar{A}_{14}=\frac{Eh^3}{128D}\frac{\left(\frac{\pi}{b}\right)^4}{\left(\frac{2\pi}{a}\right)^2\left[\left(\frac{\pi}{b}\right)^2+\left(\frac{6\pi}{a}\right)^2\right]}$$

(c) Frequency Parameter

$$\mu = \mu^{(0)} + \varepsilon^2 \mu^{(2)} + \varepsilon^4 \mu^{(4)} + \cdots$$

in which

$$\mu^{(0)} = -3\left(\frac{\pi}{a}\right)^2 \frac{b}{h}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]$$

$$\mu^{(2)} = -\frac{Eh^2}{16(1-\nu^2)} \frac{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]}{\left[4\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} \left\{(3-\nu^2)\left[16\left(\frac{\pi}{a}\right)^4 + \left(\frac{\pi}{b}\right)^4\right] + 16\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2\right\}$$

$$+ \frac{Eh^2}{8(1-\nu^2)}\left[4\left(\frac{\pi}{a}\right)^4 + \left(\frac{\pi}{b}\right)^4 + 5\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2\right] + \frac{Eh^2}{16}\left\{4\left(\frac{\pi}{a}\right)^4 + 4\left(\frac{\pi}{b}\right)^4 + \frac{\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2\right]^2}\right.$$

$$\left. + \frac{81\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2\right]^2}\right\}$$

$$\mu^{(4)} = \frac{Eh^2}{16}\left\{\frac{3\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]\left[16\left(\frac{\pi}{a}\right)^4 A_{14} + \left(\frac{\pi}{b}\right)^4 \bar{A}_{14}\right]}{\left[\left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} - \left(\frac{\pi}{a}\right)^4 (8A_{14} + 4A_{166})\right.$$

$$- \left(\frac{\pi}{b}\right)^4 (3A_{162} + A_{168}) + \frac{9\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2\right]^2}(-2A_{14} + 25A_{162} + 9A_{164} - 25A_{166})$$

$$\left. + \frac{\left(\frac{\pi}{a}\right)^4\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{3\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2\right]^2}(-50A_{14} - 49A_{168} - 49A_{166} + A_{169})\right\}$$

(ii) Vibration Mode p = 2, q = 1

(a) Deflection

$$w = w^{(0)} + \varepsilon^2 w^{(2)} + \varepsilon^4 w^{(4)} + \cdots$$

in which

$$w^{(0)} = \sin\frac{2\pi}{a}x \, \sin\frac{\pi}{b}y$$

$$w^{(2)} = 3\left[A_{14} \sin\frac{2\pi}{a}x \, \sin\frac{3\pi}{b}y + \bar{A}_{14} \sin\frac{6\pi}{a}x \, \sin\frac{\pi}{b}y\right]$$

$$w^{(4)} = 5\Big[A_{151}\sin\frac{2\pi}{a}x\sin\frac{3\pi}{b}y + \bar{A}_{151}\sin\frac{6\pi}{a}x\sin\frac{\pi}{b}y + A_{152}\sin\frac{6\pi}{a}x\sin\frac{3\pi}{b}y$$

$$+ A_{153}\sin\frac{10\pi}{a}x\sin\frac{\pi}{b}y + \bar{A}_{153}\sin\frac{2\pi}{a}x\sin\frac{\pi}{b}y + A_{154}\sin\frac{10\pi}{a}x\sin\frac{3\pi}{b}y$$

$$+ \bar{A}_{154}\sin\frac{6\pi}{a}x\sin\frac{5\pi}{b}y\Big]$$

$$+ 2\lambda_2' h\left\{\frac{\left[\left(\frac{2\pi}{a}\right)^2-3\left(\frac{\pi}{b}\right)^2\right]A_{14}}{\left(\frac{2\pi}{b}\right)^2\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{3\pi}{b}\right)^2\right]D}\sin\frac{2\pi}{a}x\sin\frac{3\pi}{b}y + \frac{\left[\left(\frac{\pi}{b}\right)^2-3\left(\frac{2\pi}{a}\right)^2\right]\bar{A}_{14}}{\left(\frac{4\pi}{a}\right)^2\left[\left(\frac{\pi}{b}\right)^2+\left(\frac{6\pi}{a}\right)^2\right]D}\sin\frac{6\pi}{a}x\sin\frac{\pi}{b}y\right\}$$

with

$$\lambda_2' = \frac{Eh^2}{16(1-\nu^2)}\frac{1}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]}\left\{(3-\nu^2)\left[\left(\frac{2\pi}{a}\right)^4+\left(\frac{\pi}{b}\right)^4\right]+4\nu\left(\frac{2\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2\right\}$$

$$A_{151} = (A_{51})_{\substack{m=2\\n=1}} \qquad \bar{A}_{151} = (\bar{A}_{51})_{\substack{m=2\\n=1}}$$

$$A_{152} = (A_{52})_{\substack{m=2\\n=1}}$$

$$A_{153} = (A_{53})_{\substack{m=2\\n=1}} \qquad \bar{A}_{153} = (\bar{A}_{53})_{\substack{m=2\\n=1}}$$

$$A_{154} = (A_{54})_{\substack{m=2\\n=1}} \qquad \bar{A}_{154} = (\bar{A}_{54})_{\substack{m=2\\n=1}}$$

(b) Membrane Stresses

$$t_{ij}' = \varepsilon t_{ij}^{(1)} + \varepsilon^3 t_{ij}^{(3)} + \cdots$$

in which

$$t_{xx}^{(1)} = \frac{Eh}{4(1-\nu^2)}\left[\left(\frac{2\pi}{a}\right)^2+\nu\left(\frac{\pi}{b}\right)^2-(1-\nu^2)\left(\frac{2\pi}{a}\right)^2\cos\frac{2\pi}{b}y\right]$$

$$t_{yy}^{(1)} = \frac{Eh}{4(1-\nu^2)}\left[\left(\frac{\pi}{b}\right)^2+\nu\left(\frac{2\pi}{a}\right)^2-(1-\nu^2)\left(\frac{\pi}{b}\right)^2\cos\frac{4\pi}{a}x\right]$$

$$t_{xy}^{(1)} = 0$$

$$t_{xx}^{(3)} = Eh\left(\frac{2\pi}{a}\right)^2\left\{A_{14}\cos\frac{2\pi}{b}y - A_{14}\cos\frac{4\pi}{b}y - \frac{\left(\frac{\pi}{b}\right)^4}{\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}4(A_{14}+\bar{A}_{14})\cos\frac{4\pi}{a}x\cos\frac{2\pi}{b}y\right.$$

$$+\frac{(\frac{\pi}{b})^4}{[(\frac{4\pi}{a})^2+(\frac{\pi}{b})^2]^2}\bar{A}_{14}\cos\frac{8\pi}{a}x\cos\frac{2\pi}{b}y+\frac{(\frac{\pi}{b})^4}{[(\frac{2\pi}{b})^2+(\frac{2\pi}{a})^2]^2}4A_{14}\cos\frac{4\pi}{a}x\cos\frac{4\pi}{b}y\Big\}$$

$$t_{yy}^{(3)}=Eh(\frac{\pi}{b})^2\Big\{\bar{A}_{14}\cos\frac{4\pi}{a}x-\bar{A}_{14}\cos\frac{8\pi}{a}x-\frac{(\frac{2\pi}{a})^4}{[(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]^2}4(\bar{A}_{14}+A_{14})\cos\frac{4\pi}{a}x\cos\frac{2\pi}{b}y$$

$$+\frac{(\frac{2\pi}{a})^4}{[(\frac{2\pi}{b})^2+(\frac{2\pi}{a})^2]^2}A_{14}\cos\frac{4\pi}{a}x\cos\frac{4\pi}{b}y+\frac{(\frac{2\pi}{a})^4}{[(\frac{4\pi}{a})^2+(\frac{\pi}{b})^2]^2}4\bar{A}_{14}\cos\frac{8\pi}{a}x\cos\frac{2\pi}{b}y\Big\}$$

$$t_{xy}^{(3)}=Eh(\frac{2\pi}{a})(\frac{\pi}{b})\Big\{-\frac{(\frac{2\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]^2}4(A_{14}+\bar{A}_{14})\sin\frac{4\pi}{a}x\sin\frac{2\pi}{b}y$$

$$+\frac{(\frac{2\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{4\pi}{a})^2+(\frac{\pi}{b})^2]^2}2\bar{A}_{14}\sin\frac{8\pi}{a}x\sin\frac{2\pi}{b}y+\frac{(\frac{2\pi}{a})^2(\frac{\pi}{b})^2}{[(\frac{2\pi}{b})^2+(\frac{2\pi}{a})^2]^2}2A_{14}\sin\frac{4\pi}{a}x\sin\frac{4\pi}{b}y\Big\}$$

(c) Frequency Parameter

$$\mu=\mu^{(0)}+\varepsilon^2\mu^{(2)}+\varepsilon^4\mu^{(4)}+\cdots$$

in which

$$\mu^{(0)}=0$$

$$\mu^{(2)}=\frac{Eh^2}{8(1-\nu^2)}\Big\{(3-\nu^2)[(\frac{2\pi}{a})^4+(\frac{\pi}{b})^4]+4\nu(\frac{2\pi}{a})^2(\frac{\pi}{b})^2\Big\}$$

$$\mu^{(4)}=-\frac{3Eh^2}{4}\Big[(\frac{2\pi}{a})^4A_{14}+(\frac{\pi}{b})^4\bar{A}_{14}\Big]$$

(3) General Algebraic Expressions for ${}_{pq}\mu^{(o)}$ and ${}_{pq}\mu^{(2)}$

$$ {}_{pq}\mu^{(o)} = \frac{D}{h}\left[\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2\right]\left\{\left[\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2\right]-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\right\} $$

$$ {}_{pq}\mu^{(2)} = -\frac{Eh^2}{16(1-\nu^2)}\frac{\left[\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2\right]}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\left\{(3-\nu^2)\left[\left(\frac{m\pi}{a}\right)^4+\left(\frac{n\pi}{b}\right)^4\right]+4\nu\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right\} $$

$$ +\frac{Eh^2}{8(1-\nu^2)}\left\{\left[p^2m^2\left(\frac{\pi}{a}\right)^4+q^2n^2\left(\frac{\pi}{b}\right)^4\right]+\nu\left[(p^2n^2+q^2m^2)\left(\frac{\pi^2}{ab}\right)^2\right]\right\} $$

$$ +\frac{Eh^2}{16}\left\{\left[p^2m^2\left(\frac{\pi}{a}\right)^4\delta_n^q+q^2n^2\left(\frac{\pi}{b}\right)^4\delta_m^p\right]+\frac{(pn-qm)^4\left(\frac{\pi^2}{ab}\right)^4}{\left[(p+m)^2\left(\frac{\pi}{a}\right)^2+(q+n)^2\left(\frac{\pi}{b}\right)^2\right]^2}\right. $$

$$ +\frac{(pn+qm)^4\left(\frac{\pi^2}{ab}\right)^4}{\left[(p+m)^2\left(\frac{\pi}{a}\right)^2+(q-n)^2\left(\frac{\pi}{b}\right)^2\right]^2}(1+\delta_n^q)+\frac{(pn+qm)^4\left(\frac{\pi^2}{ab}\right)^4}{\left[(p-m)^2\left(\frac{\pi}{a}\right)^2+(q+n)^2\left(\frac{\pi}{b}\right)^2\right]^2}(1+\delta_m^p) $$

$$ \left.+\frac{(pn-qm)^4\left(\frac{\pi^2}{ab}\right)^4}{\left[(p-m)^2\left(\frac{\pi}{a}\right)^2+(q-n)^2\left(\frac{\pi}{b}\right)^2\right]^2}(1+\delta_m^p+\delta_n^q)\right\} $$

APPENDIX B

VIBRATIONS OF A SIMPLY SUPPORTED RECTANGULAR PLATE UNDER UNIAXIAL EDGE COMPRESSION

This problem concerns itself with the vibrations of a simply supported rectangular plate which is subjected to prescribed total edge thrusts at $x = 0$ and $x = a$. The differential equation governing the static deflection of the plate is again

$$D\Delta\Delta W - h(\lambda t^{o}_{ij} + T'_{ij})W_{,ij} = 0 \tag{B.1}$$

in which

$$t^{o}_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{B.2}$$

$$T'_{ij} = \frac{1}{2} \left\langle W_{,i}W_{,j} \right\rangle \tag{B.3}$$

The operator (B.3) is identified with the same set of equations as in the main body of the paper, except for a change in the boundary conditions. That is, instead of $U'(a,y)$ and $V'(x,b)$ vanishing, the new boundary conditions read

$$U'(a,y) = k_1 \tag{B.4}$$

$$V'(x,b) = k_2 \tag{B.5}$$

in which k_1 and k_2 are determined from

$$\int_0^b T'_{xx}(a,y)dy = 0 \tag{B.6}$$

$$\int_0^a T'_{yy}(x,b)dx = 0 \tag{B.7}$$

Alternately, the previous set of boundary conditions may be used and two uniform additional tensile stresses, one in the x direction and the other in the y direction, may be superimposed so as to satisfy Equations (B.6) and (B.7). In the dynamic case, this problem has the same differential equation and same boundary conditions as in the main body of the paper except t^o_{ij} takes the form of Equation (B.2). The method of solution for this problem is the same perturbation method presented in Chapter III and hence it is not repeated here.

The general algebraic expressions for the static deflection, the static additional membrane stresses, the load parameter and the static additional membrane displacements are as follows:

$$W = \epsilon W^{(1)} + \epsilon^3 W^{(3)} + \ldots. \qquad (B.8)$$

$$T'_{ij} = \epsilon^2 T^{(2)}_{ij} + \epsilon^4 T^{(4)}_{ij} + \ldots. \qquad (B.9)$$

$$\lambda = \lambda_o + \epsilon^2\lambda + \epsilon^4\lambda + \ldots. \qquad (B.10)$$

$$U' = \epsilon^2 \left\{\frac{h^2}{16(\frac{m\pi}{a})}[-(\frac{m\pi}{a})^2 + \nu(\frac{n\pi}{b})^2 + (\frac{m\pi}{a})^2 \cos\frac{2n\pi}{b} y]\sin\frac{2m\pi}{a} x - \frac{(\frac{m\pi}{a})^2 h^2}{8} x\right\} + \ldots. \qquad (B.11)$$

$$V' = \epsilon^2 \left\{\frac{h^2}{16(\frac{n\pi}{b})}[-(\frac{n\pi}{b})^2 + \nu(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \cos\frac{2m\pi}{a} x] \sin\frac{2n\pi}{b} y - \frac{(\frac{n\pi}{b})^2 h^2}{8} y\right\} + \ldots. \qquad (B.12)$$

in which

$$W^{(1)} = h \sin\frac{m\pi}{a} x \sin\frac{n\pi}{b} y$$

$$W^{(3)} = h(B_4 \sin\frac{m\pi}{a} x \sin\frac{3n\pi}{b} y + \bar{B}_4 \sin\frac{3m\pi}{a} x \sin\frac{n\pi}{b} y)$$

$$T^{(2)}_{xx} = -\frac{Eh^2}{8}(\frac{m\pi}{a})^2 \cos\frac{2n\pi}{b} y$$

$$T^{(2)}_{yy} = -\frac{Eh^2}{8}(\frac{n\pi}{b})^2 \cos\frac{2m\pi}{a} x$$

$$T^{(2)}_{xy} = 0$$

$$T^{(4)}_{xx} = \frac{Eh^2}{4}\left(\frac{m\pi}{a}\right)^2\left\{B_4 \cos \frac{2n\pi}{b} y - B_4 \cos \frac{4n\pi}{b} y - \frac{4\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} (B_4+\bar{B}_4)\right.$$

$$\cos \frac{2m\pi}{a} x \cos \frac{2n\pi}{b} y + \frac{\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \bar{B}_4 \cos\frac{4m\pi}{a} x \cos \frac{2n\pi}{b} y$$

$$\left. + \frac{4\left(\frac{n\pi}{b}\right)^4}{\left[\left(\frac{2n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]^2} B_4 \cos \frac{2m\pi}{a} x \cos \frac{4n\pi}{b} y\right\}$$

$$T^{(4)}_{yy} = \frac{Eh^2}{4}\left(\frac{n\pi}{b}\right)^2\left\{\bar{B}_4 \cos \frac{2m\pi}{a} x - \bar{B}_4 \cos \frac{4m\pi}{a} x - \frac{4\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} (B_4+\bar{B}_4)\right.$$

$$\cos \frac{2m\pi}{a} x \cos \frac{2n\pi}{b} y + \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{2n\pi}{b}\right)^2\right]^2} B_4 \cos \frac{2m\pi}{a} x \cos \frac{4n\pi}{b} y$$

$$\left. + \frac{4\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{2m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \bar{B}_4 \cos \frac{4m\pi}{a} x \cos \frac{2n\pi}{b} y\right\}$$

$$T^{(4)}_{xy} = \frac{Eh^2}{4}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\left\{ - \frac{4\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}(B_4+\bar{B}_4) \sin \frac{2m\pi}{a} x \sin \frac{2n\pi}{b} y\right.$$

$$+ \frac{2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{2m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2} \bar{B}_4 \sin \frac{4m\pi}{a} x \sin \frac{2n\pi}{b} y + \frac{2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{2n\pi}{b}\right)^2\right]^2} B_4$$

$$\left. \sin \frac{2m\pi}{a} x \sin \frac{4n\pi}{b} y\right\}$$

$$\lambda_o = \frac{D}{h}\left[\frac{(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2}{(\frac{m\pi}{a})}\right]^2$$

$$\lambda_2 = \frac{Eh^2}{16}\left[\frac{(\frac{m\pi}{a})^4+(\frac{n\pi}{b})^4}{(\frac{m\pi}{a})^2}\right]$$

$$\lambda_4 = -\frac{3Eh^2}{16}\left[\frac{(\frac{m\pi}{a})^4 B_4 + (\frac{n\pi}{b})^4 \bar{B}_4}{(\frac{m\pi}{a})^2}\right]$$

$$B_4 = \frac{3(1-\nu^2)(\frac{m\pi}{a})^4}{64(\frac{n\pi}{b})^2[(\frac{m\pi}{a})^2+5(\frac{n\pi}{b})^2]}$$

$$\bar{B}_4 = \frac{6(1-\nu^2)(\frac{n\pi}{b})^4}{64[9(\frac{m\pi}{a})^4-(\frac{n\pi}{b})^4]}$$

Note that λ now becomes the ratio of the prescribed compression to that required for the initial instability. Let ζ be the ratio of the edge displacement caused by the prescribed edge compression to that required for the initial instability; then ζ is related to λ by

$$\zeta = \frac{\lambda u^o + \frac{1}{b}\int_o^b U'(a,y)dy}{\lambda_o u^o} \tag{B.13}$$

The λ versus ζ curve is now the load-shortening curve. The lowest buckling mode is given by $m = a/b$ and $n = 1$.

The general algebraic expressions for the deflection and the frequency parameter of the vibration mode $p = q = 1$ about the buckled configuration $m = n = 1$ are

$$w = h \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y + \epsilon^2 3h(B_{14}\sin\frac{\pi}{a}x \sin\frac{3\pi}{b}y + \bar{B}_{14}\sin\frac{3\pi}{a}x \sin\frac{\pi}{b}y)$$

$$+ \ldots.$$

$$\mu = \epsilon^2 \frac{Eh^2}{8}[(\frac{\pi}{a})^4 + (\frac{\pi}{b})^4] - \epsilon^4 \frac{3Eh^2}{4}[(\frac{\pi}{a})^4 B_{14} + (\frac{\pi}{b})^4 \bar{B}_{14}] + \ldots.$$

in which

$$B_{14} = \frac{3(1-\nu^2)(\frac{\pi}{a})^4}{64(\frac{\pi}{b})^2[(\frac{\pi}{a})^2+5(\frac{\pi}{b})^2]}$$

$$\bar{B}_{14} = \frac{6(1-\nu^2)(\frac{\pi}{b})^4}{64[9(\frac{\pi}{a})^4-(\frac{\pi}{b})^4]}$$

The general expressions for the frequency parameter of the vibration modes $p = 1, q = 1$ and $p = 2, q = 1$ about the buckled configuration $m = 2, n = 1$ are

$$\begin{aligned} {}_{11}\mu = \frac{D}{h}[-3(\frac{\pi}{a})^4 + \frac{3}{4}(\frac{\pi}{b})^4 + \epsilon^2 \frac{Eh^2}{16} \{\frac{15}{4}(\frac{\pi}{b})^4 + \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2} \\ + \frac{81(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\} + \epsilon^4\frac{Eh^2}{16}\{(\frac{\pi}{a})^4(4B_{24}-4B_{66})+(\frac{\pi}{b})^4 (\frac{3}{4}\bar{B}_{24}+3B_{62}-3B_{68}) \\ + \frac{9(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-2B_{24}+25B_{62}+9B_{64}-25B_{66}) \\ + \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-50B_{24}-49B_{68}-49B_{66}+B_{69})\} + \ldots. \end{aligned}$$

$${}_{21}\mu = \epsilon^2 \frac{Eh^2}{8}[(\frac{2\pi}{a})^4 + (\frac{\pi}{b})^4] - \epsilon^4 \frac{3Eh^2}{4}[(\frac{2\pi}{a})^4 B_{24} + (\frac{\pi}{b})^4 \bar{B}_{24}] + \ldots.$$

in which

$$B_{24} = \frac{3(1-\nu^2)(\frac{\pi}{a})^4}{4(\frac{\pi}{b})^2[4(\frac{\pi}{a})^2+5(\frac{\pi}{b})^2]}$$

$$\bar{B}_{24} = \frac{3(1-\nu^2)(\frac{\pi}{b})^4}{32[144(\frac{\pi}{a})^4-(\frac{\pi}{b})^4]}$$

$$B_{62} = \frac{3(1-\nu^2)}{4[48(\frac{\pi}{a})^4-2(\frac{\pi}{b})^4]}\left\{-3(\frac{\pi}{b})^4 - \frac{225(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]}\right\}$$

$$B_{64} = \frac{-3(1-\nu^2)}{4[48(\frac{\pi}{a})^4 + 144(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + 78(\frac{\pi}{b})^4]}\left\{\frac{81(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\right\}$$

$$B_{66} = \frac{3(1-\nu^2)}{4[16(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + 80(\frac{\pi}{b})^4]}\left\{\frac{225(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{49(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + 4(\frac{\pi}{a})^4\right\}$$

$$B_{68} = \frac{3(1-\nu^2)}{4[528(\frac{\pi}{a})^4 - 6(\frac{\pi}{b})^4]}\left\{\frac{49(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + 3(\frac{\pi}{b})^4\right\}$$

$$B_{69} = \frac{-3(1-\nu^2)}{4[528(\frac{\pi}{a})^4 + 400(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + 74(\frac{\pi}{b})^4]}\left\{\frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{3\pi}{a})^2+(\frac{2\pi}{b})^2]}\right\}$$

The general expressions for the frequency parameter of the vibration modes $p = 1$ $q = 1$, $p = 2$ $q = 1$, $p = 2$ $q = 2$ and $p = 3$ $q = 1$ about the buckled configuration $m = 3$, $n = 1$ are

$${}_{11}\mu = \frac{D}{h}[-8(\frac{\pi}{a})^4 + \frac{8}{9}(\frac{\pi}{b})^4] + \epsilon^2\,\frac{Eh^2}{16}\left\{\frac{35}{9}(\frac{\pi}{b})^4 + \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{2\pi}{a})^2+(\frac{\pi}{b})^2]^2} + \frac{16(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}\right\}$$

$$+ \;\ldots\ldots$$

$$_{21}\mu = \frac{D}{h}[-20(\frac{\pi}{a})^4 + \frac{5}{9}(\frac{\pi}{b})^4] + \epsilon^2\frac{Eh^2}{16}\{(\frac{641}{128} - \frac{4}{9})(\frac{\pi}{b})^4 + \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{4\pi}{a})^2+(\frac{2\pi}{b})^2]^2}$$

$$+ \frac{625(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{2\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\} + \epsilon^4\frac{Eh^2}{16}\{(\frac{\pi}{a})^4(72B_{34}-36B_{166})+(\frac{\pi}{b})^4(\frac{4}{3}\bar{B}_{34}+3B_{162}-3B_{167})$$

$$+ \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-450B_{34}-1225B_{166}+1225B_{166}-625B_{164})$$

$$+ \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}(-162B_{34}-121B_{166}-121B_{167}-B_{169})\} + \ldots.$$

$$_{22}\mu = \frac{D}{h}[-20(\frac{\pi}{a})^4 + 24(\frac{\pi}{a})^2(\frac{\pi}{b})^2 - \frac{140}{9}(\frac{\pi}{b})^4] + \epsilon^2\frac{Eh^2}{16}\{-36(\frac{\pi}{a})^4 - \frac{4}{9}(\frac{\pi}{b})^4$$

$$+ \frac{256(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{3\pi}{b})^2]^2} + \frac{4096(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{\pi}{b})^2]^2} + \frac{4096(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{3\pi}{b})^2]^2} + \frac{256(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{\pi}{b})^2]^2}\}$$

$$+ \ldots.$$

$$_{31}\mu = \epsilon^2\frac{Eh^2}{8}[(\frac{3\pi}{a})^4+(\frac{\pi}{b})^4] - \epsilon^4\frac{3Eh^2}{4}[(\frac{3\pi}{a})^4B_{34} + (\frac{\pi}{b})^4\bar{B}_{34}] + \ldots.$$

in which

$$B_{34} = \frac{243(1-\nu^2)(\frac{\pi}{a})^4}{64(\frac{\pi}{b})^2[9(\frac{\pi}{a})^2 + 5(\frac{\pi}{b})^2]}$$

$$\bar{B}_{34} = \frac{3(1-\nu^2)(\frac{\pi}{b})^4}{32[729(\frac{\pi}{a})^4-(\frac{\pi}{b})^4]}$$

$$B_{162} = \frac{3(1-\nu^2)}{4[132(\frac{\pi}{a})^4 - \frac{4}{3}(\frac{\pi}{b})^4]} \{- \frac{1225(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} - 3(\frac{\pi}{b})^4\}$$

$$B_{164} = \frac{3(1-\nu^2)}{4[132(\frac{\pi}{a})^4 + 256(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + \frac{236}{4}(\frac{\pi}{b})^4} \{\frac{625(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\}$$

$$B_{166} = \frac{3(1-\nu^2)}{4[64(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + 80(\frac{\pi}{b})^4]} \{36(\frac{\pi}{a})^4 + \frac{1225(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + \frac{121(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\}$$

$$B_{167} = \frac{3(1-\nu^2)}{4[3540(\frac{\pi}{a})^4 - \frac{20}{3}(\frac{\pi}{b})^4]} \{\frac{121(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2} + 3(\frac{\pi}{b})^4\}$$

$$B_{169} = \frac{3(1-\nu^2)}{4[3540(\frac{\pi}{a})^4 + 1024(\frac{\pi}{a})^2(\frac{\pi}{b})^2 + \frac{220}{3}(\frac{\pi}{b})^4]} \{- \frac{(\frac{\pi}{a})^4(\frac{\pi}{b})^4}{[(\frac{5\pi}{a})^2+(\frac{2\pi}{b})^2]^2}\}$$

For the cases of $a = b$ and $a = 2b$ results are shown in Figures 9 and 10.

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